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Triangle similarity: interactions in meshes and slider

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Abstract: This article discusses the interactions of (prospective) teachers when performing two tasks in a multiuser, online, virtual, and synchronous version of GeoGebra, the VMTcG. When analyzing the data, the discursive aspects, the modalities of dragging points and the categories of signs present in the process of semiotic mediation are considered. The first episode involves the use of the square grid in a task on congruence of triangles, and the second one, a task on the similarity ratio from a slider. The data make it possible to conclude that the thought processes related to the use of the checkered grid were limited to the exploration of more global properties and relations of ascending aspect, while those related to the use of the slider may represent a more potent form of thought process, which implies both the global observation of properties and the validation of conjectures.

Keywords: Similarity. Congruence. Triangles. Proportionality. VMTwG.

Similitud de triángulos: interacciones en mallas y control deslizante

Resumen: Este artículo analiza las interacciones de (futuros) docentes al realizar dos tareas en una versión multiusuario, en línea, virtual y síncrona de GeoGebra, el VMTcG. Al analizar los datos, se consideran los aspectos discursivos, las modalidades de los puntos de arrastre y las categorías de signos presentes en el proceso de mediación semiótica. El primer episodio involucra el uso de malla a cuadros en una tarea sobre congruencia de triángulos, y el segundo, una tarea sobre la relación de similitud de un control deslizante. Los datos nos permiten concluir que los procesos de pensamiento relacionados con el uso de la malla a cuadros se limitaron a la exploración de propiedades y relaciones más globales de aspecto ascendente, mientras que los relacionados con el uso del control deslizante pueden representar una forma más potente de proceso de pensamiento, que implica tanto la observación global de propiedades como la validación de conjeturas.

Palabras clave: Semejanza. Congruencia. Triangulos. Proporcionalidad. VMTcG.

Semelhança de triângulos: interações em malhas e controle deslizante

Resumo: Neste artigo discute-se sobre as interações de (futuros) professores ao realizar duas tarefas em uma versão multiusuário, *on-line*, virtual e síncrona do GeoGebra, o VMTcG. Ao analisar os dados, consideram-se os aspectos discursivos, as modalidades de arrastar pontos e as categorias de signos presentes no processo de mediação semiótica. O primeiro episódio envolve o uso da malha quadriculada em uma tarefa sobre congruência de triângulos, e o segundo, uma tarefa sobre a razão de semelhança a partir de um controle deslizante. Os dados possibilitam concluir que os processos de pensamento relacionados ao uso da malha quadriculada limitaram-se à exploração de propriedades e relações mais globais de aspecto

ascendente, enquanto os relativos ao uso do controle deslizante podem representar uma forma mais potente de processo de pensamento, que implica tanto a observação global de propriedades como a validação de conjecturas.

Palavras-chave: Semelhança. Congruência. Triângulos. Proporcionalidade. VMTcG.

1 Introduction

In the year 2020, humans witnessed a new moment in history due to the Covid 19 pandemics, and quite a few ways to communicate, learn and teach which already existed became more necessary because of physical distancing.

Emergency remote teaching[1] led to rethink the possibility to learn and teach through distance and with the use of devices connected to internet, and researchers delved into virtual and synchronous environments. This study was motivated by the need for new paths in the teaching and learning geometry, specifically triangle similarity, trying to understand what contributions and synchronous interactions in Virtual Math Teams (VMT) environment can be observed in the learning of (prospective) teachers. We use the expression "(prospective) teachers" to indicate the inclusion of both graduate and undergraduate teachers in our working team. VMT allows for the creation of workspaces using Desmos or GeoGebra, and the latter is called Virtual Math teams with GeoGebra (VMTwG).

VMT adds the Dynamic Geometry Environment (DGE) GeoGebra and makes it possible to drag points, free or linked to a figure, allowing for the production of hypotheses on the observed properties (Arzarello, Olivero, Paola & Robutti, 2002), as well as for the verification of relations among objects and the dynamics of relations (Alqahtani & Powell, 2016). As it is a multiuser environment, the VMT constitutes collaborating groups that explore, question, and learn together (Stahl, Koschmann & Suthers, 2008), involving the interaction between the individual and the device as well as among the subjects. We consider not only the importance of an individual's learning, his or her mind with itself, but also the possibility of collective learning.

Similarly, we understand that tasks in a DGE open the way to the production of meanings, whether personal or mathematical, although not automatically (Bussi & Mariotti, 2008). It becomes necessary, therefore, to have a didactic project that supports the students in these environments. In this way, we seek to understand the role of the checkered grid, the task script, and the chat in a task of geometry, virtual and synchronous, in the configuration of thinking among teachers and prospective teachers. We would like to stress that a checkered grid /mesh is a resource available at the graphic zone of VMTwG, the chat is one of the spaces of VMTwG designed to exchange posting, and the script for the task was made by the authors (Brito, 2022).

This article discusses the empirical aspects of the research leading to a master's degree for Brito (2022)[2], and it aims to study the synchronous interactions of (prospective) teachers in online tasks on triangle similarities and establish the role of the checkered mesh and sliding control in the configuration of geometric thinking among the participants. The theoretical framework for our literature review that helped us with the analyses of generated data of VMT is explained in the next section below.

[1] Resolution CNE/CP nº 2, Dec 10, 2020. Available at: https://www.in.gov.br/en/web/dou/-/resolucao-cne/cp-n-2-de-10-de-dezembro-de-2020-293526006, access Jan 1, 2023.

[2] This article issues from an MA thesis in Science and Mathematics Education (PPGEduCIMAT) at Universidade Federal Rural do Rio de Janeiro (UFRRJ), by Cristiano Brito, under the direction of Marcelo Bairral.

2 Similarity, DGE and VMTwG: some previous studies

The similarity of plane figures is usually related to the notion of measure, amplification, and reduction, keeping the proportion of the corresponding sides and angles.

One classical example of the application of similarity of plane figures is when you slide your fingers over the screen of a smartphone, tablet or any other such touchscreen devices, to visualize an image without deforming it. It is also usual in computer programs when you alter the length and height of figures, keeping the proportion of their corresponding sides when you drag the image diagonally.

Among other examples, similarity is also applied to fields of knowledge such as engineering, architecture, the study of optical phenomena, videogame programming and digital image processing. Such fields of study have the professional aim to control the dimensions and proportions of the shapes involved (Maciel & Almouloud, 2004; Powell & Alqahtani, 2021).

Within mathematics, the similarity of figures is related to homothety, the right triangle metric ratios, the invariance of trigonometric ratios (Jaconiano, Barbosa, Concordido & Costa, 2019; González *et al*., 1990), the demonstration of mathematic theorems, as the fundamental theorem of proportionality and the Thales theorem (Pereira, 2017). It is in this sense that Lima (2011, p. 31) comments that "the concept of similitude, mainly of triangles, is quite prominent. Books usually define similar triangles as those that have 'equal angles and proportional homologous sides'. This definition is extended to polygons".

Triangles are the only group of non-deformable polygons, and "the two conditions that guarantee similitude always take place together […]: if the sides are proportional, the corresponding angles will automatically be equal, and vice-versa" (Machado, 2000, p. 25). The triangles show specific characteristics of similitude, which constitute the basis to understand the similitude of other polygons.

González *et al*. (1990) highlight that the criteria that establish the similitude of triangles are not well established for the learning subjects. According to them, there is some language interference of mathematical language into everyday language, because the word "similar" has different meanings depending on the type of context of the language being used. We have specified the way mathematics defines similar figures. Nevertheless, in everyday language, similar objects can be either equal or only resembling each other. We observe that from a mathematics point of view, we are dealing with something clearly more defined: figures that have the same shape, with equal or different sizes (Pereira, 2017).

González *et al*. (1990, p. 148) stress this about those two criteria, equal corresponding angles and proportional homologous sides:

> The first criterion is probably the easiest to identify, it is obvious, but the second criterion, the similarity ratio between the figures is a concept that is closely related to the proportionality between quantities. It needs to be worked upon with plenty of activities significant enough to ensure their acquisition.

Based on the mapping (Brito, 2022) developed in the research methodology, we identified some papers that narrate the way the learners and (prospective) teachers apply the concepts of similarity and proportionality in mathematical problems.

Galvão, Souza and Miashiro (2016), for example, developed some work on trigonometric functions among undergraduate students of mathematics. These authors observed that the students' lack of mastery of trigonometry when dealing with right triangles, Pythagoras theorem, measuring angles (in degrees and radians) and in the definition of the sine function was all a result of the little understanding they had about triangle similitude.

Other studies involving mathematics teachers stress the abusive use of the rule of three as a method to solve geometry problems, without considering concepts of proportionality and similarity (Costa & Allevato, 2015; Tinoco, 1996). Along these lines, it is argued that the teaching of similarity should not be restricted to how to solve something, but rather stress on the reason why one method is used, so as to avoid prejudice when learning (Jaconiano et al., 2019; Menduni-Bortoloti & Barbosa, 2018).

We understand that this scenario brings about the need for new practices in the teaching of triangle similarity, to enable the development of geometric and proportional thinking among learners. The concept of proportionality is very important to make connections, as it is an integrating content of different branches of mathematics (Tinoco, 1996). And, more than a content to be taught, proportionality works in building cognitive structures that are necessary for the comprehension of other mathematics concepts, whether in the numeric or the geometric field (Costa & Allevato, 2015). In this sense, González et al. (1990) propose working the concept of similitude from figure enlarging and reduction and by observing variant and invariant characteristics.

Based on the dialogic notion of mathematics in Gattegno (1987, p.10), we understand that the mathematics activity consists of a "dialog of a person's mind with itself" and goes beyond the usual talk to oneself, although it is related to that notion. Nevertheless, such a notion is more specific about the existence of a consciousness of human activity, practiced on what we call science. In this way, the author considers (p. 16) mathematics as one of the achievements of the human mind acting on itself, one of the clearest dialogs of the mind with itself. Mathematics is created by mathematicians who, first, talk with themselves and, later, to one another (Gattegno, 1987).

For Gattegno (1987), mathematic activity takes place in the dialog of a mind with itself over observations of objects, relations among objects, and relations among relations or dynamics. We consider that mathematic activity of (prospective) teachers regarding triangle similitude takes place from dialogs with themselves over observations of objects (angles, sides and their measurements), relations among objects (corresponding angles, sides and points) and relations among relations or dynamics (congruence of corresponding angles, ratio, proportion and proportionality of homologous sides, as well as similitude among triangles).

Based on the literature regarding the subject, we consider important to establish a new approach on the similitude of triangles from the use of DGE with the audience of (prospective) teachers. The logic involved in a DGE emphasizes the development of the mathematic activity through moving or deformable figures, as a resource to promote exploration, discovery, and research of mathematic objects, whether through software or not (Arceo[3], 2009 apud Bairral & Barreira, 2017, p. 47). It is important to highlight that those processes do not occur automatically. They require the teacher's mediation while the actions are taking place in the DGE (Arzarello et al., 2002).

Among the particularities of a DGE through software, we are stressing the possibility to drag free points and transforming figures, keeping or not keeping Euclidian properties (Bairral & Barreira, 2017). The dragging action helps the individual to produce conjectures through the observation and exploration of the movements of a figure and through the discovery of invariant

[3] Arceo, E. D. B. (2009). *Geometría dinámica con Cabri-Géomètre*. (3. ed.). Metepec: Editorial Kali.

properties (Azarello et al, 2002).

Those features can be enhanced by making groups of interaction in virtual environments, working on the resolution of mathematic tasks in a collaborative way (Powell, 2014). It was with this aim that VTM was developed, so that participants could interact with one another through a chat, build figures and drag points in the graphic area of GeoGebra.

In this way, with GeoGebra added to VMT, besides the interaction among individual and DGE, other elements are related, like the interaction among the participants in the group that has come into play. We use the term "interaction", from discourse analysis studies, as a synonym for interlocution and meaning exchange, reflection, and negotiation of meanings, with the possibility to take place among two or more interlocutors subjected to shared norms, orally or in writing, with or without the use of technology (Maingueneau, 2004; Oliveira & Bairral, 2020).

Interaction related to the use of VMT focuses on the collaboration among individuals, so it is a collaborative interaction in which "the group works for the joint task and learning" (Oliveira & Bairral, 2020, p. 280), where the students "learn through their questions, pursuing joint lines of reasoning, teaching one another and watching how others are learning" (Stahl *et al*., 2008, p. 4). According to the same authors, this form of interaction diverges from cooperative interaction in that the latter consists in splitting the task into individual subtasks that are later put together into partial results to make a result. Collaborative interaction implies that the task is carried out jointly — even the discourse that issues from it does not belong to one individual, but to the collective that has issued from the joint work.

In addition to the particularities and potentialities of a DGE, we are considering the contexts of teacher and learner, external and internal. In the next section we are dealing with internal context, while we are dealing here with external context: the one composed of the computer with DGE, including basic functions of constructing and dragging, the signs that involve figures, the task script and even the gestures. We are considering everything that generates meaning for the subject involved as a sign.

DGE and its functions comprise a microworld in which the logic of Euclidean geometry is embedded (Mariotti, 2000) and it is possible to produce graphic strokes where the properties of the object are preserved thanks to the working of the tools that operate in this microworld. We shall call this microworld an artifact (Bussi & Mariotti, 2008), and the functions of dragging are classified as direct or indirect dragging. In direct dragging it is possible to drag a point without interfering with another, whereas in indirect dragging there is some interference in the movement of other points and in elements built from the original point.

It is worth noting that the meaning of artifact is not restricted to the DGE, but is a wider notion, including every type of human creation that has a practical nature, everything to which an individual attributes a use and which, at the same time, moves from the sphere of intellect to the sphere of practical doings and vice-versa (Bussi & Mariotti, 2008).

In the next section we shall discuss about the modalities in dragging and cognitive processes (Arzarello et al. 2002), in relation to internal context. Besides, we shall approach the different categories of signs (Bussi & Mariotti, 2008).

3 Theoretical framework

Starting from the principle that online and synchronous virtual environments can provide a favorable space for learning both individually and collectively, we understand that collaborative interactions and the role of VMT can be analyzed through discursive aspects, through interactions among individuals with DGE, through the dragging action and through the meanings that individuals apply to each form of dragging.

Arzarello et al. (2002) identified seven modalities of dragging in implementations with learners who used Cabri-Géomètre. The types of dragging identified are random dragging, bound dragging, guided dragging, dragging to the fictitious lotus, dragging linked in a curve and dragging test. These modalities are related to the type of ascending cognitive process, going from construction to theory, or descending process, going from theory to construction. Besides, these authors relate the objective through which the individual drags one given point to the phases of resolution of a problem, which are: discovery phase, construction of a conjecture, and validation of a conjecture.

These same authors point out that the dragging modalities can and must be inserted into the culture of the classroom, so that they are available to the students. Chart 1 synthetizes the dragging modalities proposed by Arzarello et al. (2002) and relates the phases of resolution for each modality and the aims for which they are used and defines each modality according to either ascending or descending cognitive process.

Chart 1: Synthesis of dragging modalities and cognitive processes involved

Phases in the resolution of the problem	Modalities of dragging	Characterization	Goal aimed with its use
Discovery phase	Random dragging	Moves free points in a random way to discover the regularities of the built figure.	To explore a given task (ascending control).
	Bound dragging	Like the function of Random dragging, this modality only applies to the points linked to the figure (semi-draggable points).	To explore a given task (ascending control).
	Guided dragging	Dragging free points in the figure to give it a specific shape.	To explore a given task (ascending control).
Construction of the conjecture	Dragging to a fictional *locus*	Dragging is more intentional and allows for the discovery of some regularities of the figure, although local properties are still not explicit, as they remain in a fictional *locus*.	To produce new heuristics and /or to organize the logic of previous research (Transition from ascending to descending control).
	Linked dragging	The subject links one point to the geometric desired place and drags the point in order to maintain the discovered property.	The fictional *locus* becomes visible. (Transition from ascending to descending control).
Validation of a conjecture	Dragging in a curve	New points are marked along a curve in order to maintain the regularity of the figure.	Descending control
	Dragging test	The individual drags free points and linked points to objects in order to verify the permanence of initial properties.	Descending control

Source: Own elaboration based on Arzarello *et al*. (2002)

In a practical way, in an activity with DGE, understanding the dragging modalities can provide (prospective) teachers with tools to analyze cognitive processes of the learners, following the moments of resolution of a problem, which can be identified by the type of dragging being used (Arzarello et al., 2002).

The discovery phase comprises random dragging, boundary dragging and guided dragging. In this phase, learners drag with the purpose of exploring a figure in order to understand its functioning. These modalities are part of the ascending control flux, going from the construction to the theory, through an exploration free of regularities, invariants, among other elements that belong to objects (Arzarello et al., 2020).

The dragging to the fictional locus and the linked dragging represent more intentioned ways to drag and can compose a crucial moment of change from ascending to descending control. Researchers Alqahtani & Powell (2016, 2017b) organized a training course for teachers of mathematics, online and synchronously on VMTwG, and they noticed that the dragging resource was used more knowingly, to the extent that teachers didn't only check the features contained in the construction, but they also checked the validity of the construction.

The tasks proposed by Alqahtani & Powell (2016, 2017b) involved triangles that were dependent on one another, where indirect dragging was necessary for the resolution of the problem that was proposed. The teachers participating in the research used the dragging resource together with the algebra window, aiming to understand the dependence of measures of the angles and the sides. Besides, the teachers also "dragged a triangle on top of another, or whatever they called overlapping triangles… so that in almost every task after this one, they discussed some form of congruence and similitude of objects" (Alqahtani & Powell, 2017b, p. 32).

The authors stress the importance of educators' understanding how their students take over virtual collaborating environments, and how this appropriation brings a new configuration to the geometric thinking of the learners. These authors highlight two aspects in tasks with DGE: the manipulation of geometric objects by resorting to dragging and the discourse based on the relations and dependencies observed when dragging the objects. The resource of dragging is related to the external context of the student, that is, to the DGE itself and to the icons that allow the manipulation of objects. The dragging action is interlinked with the students' discourse, as it opens the way to actions that can become part of the mathematic discourse, thinking and communication of geometric ideas (Alqahtani & Powell, 2017b).

It is in this sense that Bussi & Mariotti (2008) discuss the role of artifacts and of signs in a given mathematic task, comprising the external context (the artifact) and the internal one (the signs). Artifacts act externally and are related to their practical perspective in the task, that is, the way in which the individual attributes them a use. For instance, there are situations which come into play in the use of a hammer. Depending on what it is aimed for, it can be used to hammer a nail into the wall or to take it out. It will be necessary to consider the amount of strength, considering the position. Situations like the ones regarding the use of the hammer make it an artifact. We shall now move on to deal with the internal context.

Besides the physical device, an artifact such as the VMT, through its dragging function, can internally guide the behavior of an individual, that is, it can affect his or her cognitive activity. Despite differences between them, the sign and the artifact have in common the function of mediation in the resolution of a task, but they differ in the way they orient human behavior. Vigotski calls "internalization" the process "in which individuals transform external activities linked to artifacts into internal activities which are linked to signs" (Alqahtani & Powell (2017b, p.3), involving "the inner reconstruction of an external operation" (Bussi & Mariotti, 2008, p. 751).

Internalization is led by semiotic processes. In other words, this means that the process involves a semiotic mediation that consists not only in encouraging the relation of mathematic knowledge with the student, but also comprises the links between signs and the content being studied. According to Bussi & Mariotti (2008), there is a system of signs of the artifact, that relate the artifact to the specific task that is aimed at these signs have a strong link to the procedures that are carried out, and the semiotic means through which these signs are produced go from gestures, words, or figures.

The second system of signs -called mathematic signs- is parallel to the one mentioned above, and it consists of the relation between an artifact and mathematic knowledge. This relationship is expressed through signs that reveal properties embedded in the artifact. These two systems do not relate spontaneously, and that is why it is important for the teacher to understand the evolution of signs, from artifact to mathematics. Usually, the teacher can explore the signs elaborated socially, aiming to guide the evolution of signs towards what is known as mathematics, relating personal meanings generated using the artifact to mathematic meanings (Bussi & Mariotti, 2008).

In general terms, the teacher who masters both personal and mathematic meanings can orient the evolution of the signs in the context of the artifact, relating to the artifact and to experience for its use, for mathematic signs, as definitions, conjectures to be proven or mathematic proof.

There is a category of signs, described by Bussi & Mariotti (2008) as pivot signs. They work as pivots or hinges, and they allow for the signs of the context of the artifact to evolve towards the mathematic realm. These signs belong to the mathematic field as much as they belong to the artifact. As what is at play are signs, the teacher can use the pivot signs to relate the personal meanings of his or her students, leading them to mathematic meanings. This double relation is called “semiotic potential” and does more than relate mathematic knowledge to students. It demands a system of transformations of signs into other signs. It is therefore not just a mediation process; it is semiotic mediation.

The artifact also plays a role of semiotic mediation, although not in an automatic way. The teacher’s guiding is necessary for the evolution of signs with a stress on the use into signs in the mathematic context. It is worth noting that “any artifact will be referred to as a tool for semiotic mediation, if it is used (or it is meant to be used) intentionally by the teacher to mediate a mathematic content through a planned didactic intervention.” (Bussi & Mariotti, 2008, p. 754).

In this section, we have presented the theoretical studies that oriented our research. We consider that they can contribute to the teaching practice in following and analyzing the cognitive processes of students as they interact with both the artifact and with other individuals. In the next section, we are presenting the methodological aspects of our research.

4 Methodological Approach

The acquaintance with VMT arose in the dialog between the researcher and his tutor from the opportunity to implement tasks that had been planned before the pandemics but could only be carried out during the confinement time. It is worth noting that, initially, we were planning to apply a didactic sequence on the similitude of triangles, with the use of GeoGebra for smartphones, to learners in their 8th year of elementary education.

Due to the duration of the pandemics and the need to have emergency remote teaching, without knowing when it would be possible to return to actual classroom activities, it was possible to carry on our research with the participation of undergraduate (prospective mathematics teachers) and graduate (master) students in program PPGEduCIMAT. Both groups were following online courses of Subject *Teaching and Learning Mathematics in Virtual Environments* in the year 2020, and this was their first experience with VMTwG and the activities it proposed.

This was a time for self-knowledge for the author as a teacher and a researcher, as he had to define new directions for his work, and he got to know VMT. It is important to stress that the need for distancing and the insertion in VMT meant a challenge to rethink the role and participation of teachers in interaction and in the mathematic discourse in class. We also looked for ways to understand in which ways learners construct their knowledge both individually and collectively, paying special attention to dragging and group interaction.

We went on to do the necessary adaptations to the tasks that had initially been thought for off-line situations with the use of smartphones for on-line and synchronous tasks. The VMT allows the user to create independent rooms, or courses organized with rooms organized in groups. The working space can be created for the use of Desmos or GeoGebra. Following this, the activities were planned to take place in an average time of two hours and with the participation of three to four users per room. We chose this organization mode to favor understanding among participants during a high flow of data generated in the area of the chat.

The VMTwG interface comprises: GeoGebra area of tools which we shall refer to as icons, GeoGebra graphic zone, algebraic zone, chat area, button to ask for the control, button to refer previous postings in the chat and area for the presentation of instructions to the task. Figure 1 presents the interface of a VMTwG room and the subareas that belong in it.

Figure 1: Subareas in VMTwG Interface

Source: Own Elaboration from VMT environment

The button to request the control allows one user at a time to use the GeoGebra icons to make constructions on the graphic area, drag points and manipulate an embedded sliding control. While this is taking place, the other participants can observe and share their ideas on the chat. In case another participant has already selected the button 'take control', the posting 'Can I take control' is generated automatically on the chat, and the candidates must wait for the control to get free.

All the data generated by the functions of a VMT room are saved in a sort of cloud on the site, which can be accessed later, and the data can be reviewed in chronological order. This resource called *Replayer* allowed for the analysis of the data of the implementations carried out in the research done by Brito (2022), even with data in charts, graphs, and filters generated by the system itself. The editing functions of the room are available only to the teacher /monitor of the room, and he or she can insert the statement of a problem in the area meant for the instructions for the task.

We moved on from a didactic sequence originally planned to take place in seven meetings off-line, to a didactic sequence consisting of three tasks on-line and synchronous, with the following learning aims:

- *Task 1* — research the concept of congruence of triangles by the overlapping of two triangles and identifying the corresponding angles and sides.
- *Task 2* — make out variation and covariation factors of two similar triangles, by comparing the ratio of their corresponding sides.
- *Task 3* — check necessary and sufficient conditions for two triangles to be similar, from the exploration of the case of similitude of triangles Angle-Angle (AA).

In each proposed activity, we tried to name the VMTwG rooms without mentioning the concept of congruence or similitude of triangles, to enable the production of argumentation and justification without direct interference, only through directions in the statement of the task and, in some cases, with the interference of the teacher mediating in the room. The tasks are closed in nature, that is, they are focused on the procedures of construction, the involved properties in the construction, and the concepts that are already known by the participants (Barreira & Bairral, 2017).

Of the three tasks, we concentrated only in the first and second one, as they involve two resources used by some teachers in tasks like GeoGebra: the checkered mesh and the sliding control. In the next section, we are presenting two situations that comprise the analysis of the data and we are discussing the theoretical framework based on an online and synchronous task developed with prospective mathematics teachers about the concept of similitude of triangles.

5 Results

We based our analysis for the first task, on congruent triangles, and for the second task, on the ratio of triangle similitude, on the theoretical contributions of Arzarello et al. (2002) to understand the prospective mathematics teachers' aims when dragging free points of two triangles created with no dependence on each other and with the use of a checkered mesh.

The unit of analysis is the participants' speech, that is, the registers written by participants. From these registers which were taken out of interaction in the chat, we recovered the moments of the use of the resource, and we associated that to the registers in the chat. We looked the postings before and after the dragging action and, finally, we crossed the discursive data and graphs to interpret that action. We used the same procedure to analyze episodes 1 and 2, that is, the checkered mesh and the sliding control.

In the first episode, we are analyzing the function of the task script, the chat, and the graphic zone in the process of semiotic mediation. In the second episode, we are describing the analysis under the perspective by Bussi & Mariotti (2008) to understand the role of sliding control in the process of semiotic mediation and to identify the pivot sign and the relation between signs of the artifact and mathematic signs.

5.1 Episode 1: dragging congruent triangles over the checkered mesh.

The first task proposed the construction of two triangles over a checkered mesh, comparing the sides of triangles which had been built separately, and dragging free, non-dependent points, to form congruent triangles. This task took place in the context of a subject given by the author and undergraduate students of mathematics and graduate students in an MA program on Science and Mathematics Teaching in year 2020. The teacher mediating the room was Brito, the first author. All participants had already interacted in VMTwG in other geometry tasks and had some experience with GeoGebra. We are giving them fictional names here: Nicole, Maria and Ivone. Nicole and Maria were undergraduates and Ivone was doing studies leading to an MA.

Initially, they organized themselves so that each participant could control, one at a time, the VMT and use the construction tools. They built both triangles and, following the task script, they observed the influence of the movement of one vertex to determine the length of the other sides and they verified there was no dependence between the triangles.

Maria and Nicole wondered about the question in the script: “a) With selected tool , move freely the points of the triangle vertices and comment on what you observed”. Chart 2 shows the segment of the interaction between the participants.

Chart 2: Fragment (84-90, 98) retrieved from the *chat* of episode 1

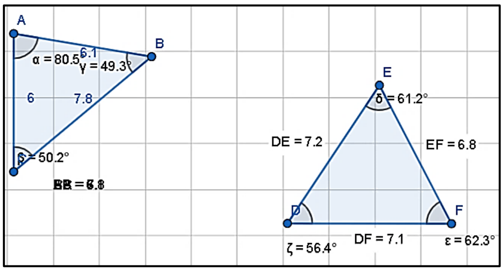
Index /pointer	Participant	Posted in the *Chat*
84	Nicole	About 5-a) both triangles have angles with different measures and different length of the sides. But if they were both in agreement with the mesh, if they were built in the same position, they would be the same.
85	Nicole	I’d like to do it, but I can’t get the control.
86	Nicole	Can I take control?
87	Maria	Then triangle DEF should have been built similar to the first?
88	Nicole	I don’t know, Maria. But if it had been built equal, they would have same angles and sides.
89	Nicole	At the moment, they are completely different triangles.
90	Maria	I got it, I thought it was supposed to be different.
98	Ivone	Question 5, letter B: Triangles ABC and DEF have their sides the same length and equal angles when you move the mesh, when observing measurement. Sides AB=DE, BC=EF and AB=DF, in this case the triangles can be scalene or isosceles.

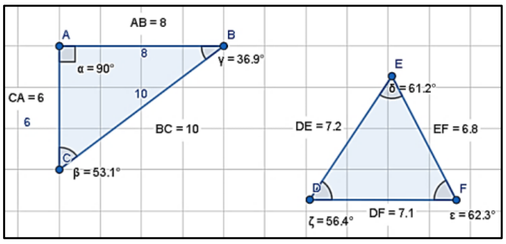
Source: Own elaboration from VMT data

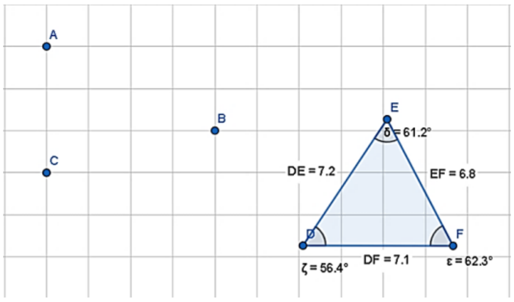
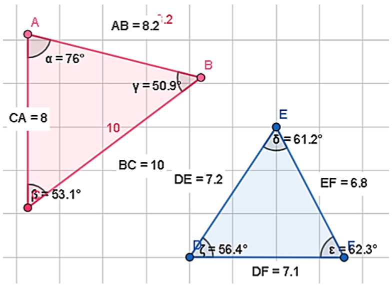
The three of them discussed over the possibility to equal the measures of sides and angles by moving points over the checkered mesh. At a certain point, they sought to interpret the aim of the task (Chart 2, index 87). Such questioning enabled interaction among participants in the exchange of information about the properties of the triangles which they called similar, with angles and sides of equal measures (Chart 2, index 88). Each participant’s remarks were shared in the chat with the other participants, and through this movement, some conjectures emerged in the search for the solution of the problem.

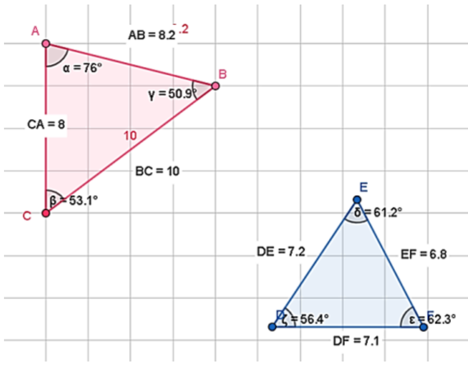
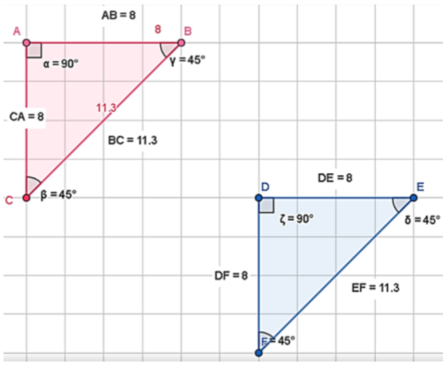
While Maria and Nicole discussed over the use of the checkered mesh to build similar triangles (Chart 2, indexes 84-90), Ivone was not manifesting herself because she was performing some changes in the graphic zone. Nevertheless, Ivone’s actions were put into practice because of the ideas shared in the chat by the other participants about the possibility to put the vertices of the triangles in the same positions of the checkered mesh and form equal triangles. Chart 3 shows the moment Ivone uses the checkered mesh following the ideas shared by Nicole and Maria.

Chart 3: Processes used by Ivone while dragging the checkered mesh.

<table>
<tr><th>Parameter</th><th>Capture of Replayer</th><th>Procedures used when dragging the mesh</th></tr>
<tr><td>1</td><td>


(A)


(B)</td><td>Ivone drags DEF and ABC without altering the measures (1A). Then she drags points A, B and C, forming a right triangle (1B).</td></tr>
<tr><td>2</td><td>

</td><td>Ivone reconstructs ABC, adding the points in the checkered mesh</td></tr>
<tr><td>3</td><td>

</td><td>Ivone drags point C and then triangle ABC, keeping the measures.</td></tr>
</table>

4		Ivone drags ABC on the mesh, keeping its measures, and drags DEF, keeping measures.
5		Ivone drags free points of triangles on the mesh to form isosceles right triangles with equal measures for the corresponding sides and angles.

Source: Own elaboration from the Replayer data.

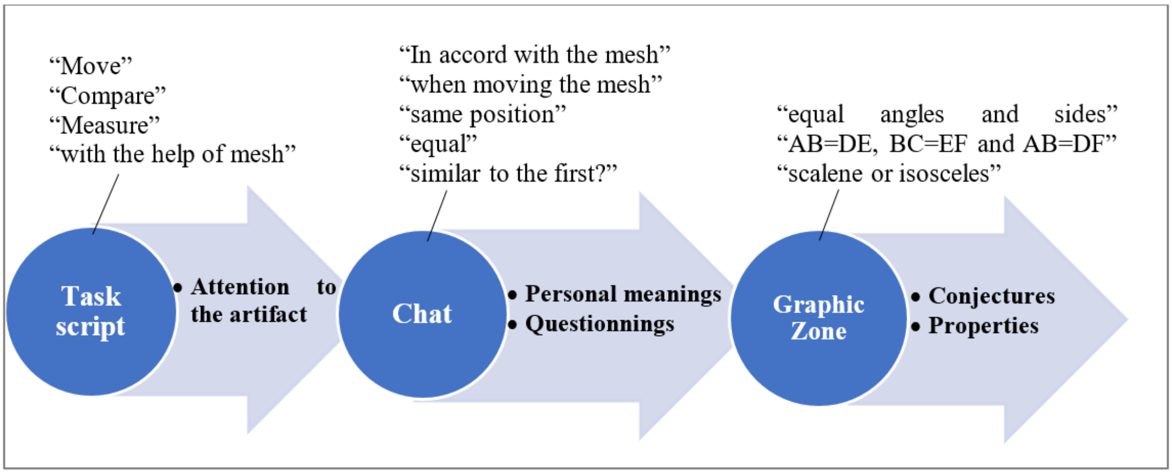
The sequence of parameters 1 to 5 presents the way in which Ivone dragged the free points of the figure over the checkered mesh to: i) form a right triangle (Chart 3, parameter 1, Figure B), ii) build triangle (Chart 3, parameters 3 and 4), and create strategies for the next move, iv) build the second isosceles right triangle congruent with the first (Chart 3, parameter 5).

Ivone's actions are related to the continuity of interactions started in the chat by the other participants, which materializes in the construction of the isosceles right triangle and the next posting (Chart 2, index 98). The information posted by Ivone confirm Nicole's conjecture, who related the position of the triangle vertices on the checkered mesh and the equal measure of the sides. Ivone still adds some information which had not been included in the chat about the types of triangles that fit the observed situation.

This segment shows the interrelation among the spaces of the VMT, the algebraic zone and the chat, as the participants' remarks and observations were applied in the graphic zone. Besides, the task script directed the attention to the aim of the task and the artifact itself, through the dragging action, generating signs related to the use of the artifact. From this use, there arose questionings that enabled the exploration of possibilities pointed out through the guided dragging, aiming to make a specific construction of isosceles right triangles. Figure 2 synthesizes the semiotic mediation process described in the analyzed episode.

We should point out that each component has its own specific role in the mediating process and all components support each other in the advancement of the participants during the task. The task script played the role of guiding the participants to the aim of the task regarding mainly the use of VMTwG through the resource of dragging the checkered mesh. This is made evident by the use we made in the script of expressions "move", "compare" and "move with the help of the mesh". The participants even used stretches of the script to solve or point out the steps that were being solved (Chart 2, index 98).

Figure 1: Semiotic Mediation Process composed of script, *chat*, and graphic zone.



Source: Own Elaboration

As the task involved the construction and manipulation of triangles with free points, during the carrying out the participants stressed the basic function of a DGE -the movement- with the possibility to build or not to build congruent triangles. As the movement of a triangle did not imply the alteration of the other one, the role of dragging was to point out the properties of each triangle individually.

The chat was used to share the personal meanings, interpret what the participants were saying and doing, inform of the observations and properties. Together with the task script, the dragging resource was used to construct some types of congruent triangles, with the use of checkered mesh, which worked as a type of comparison measure between the sides of the triangles.

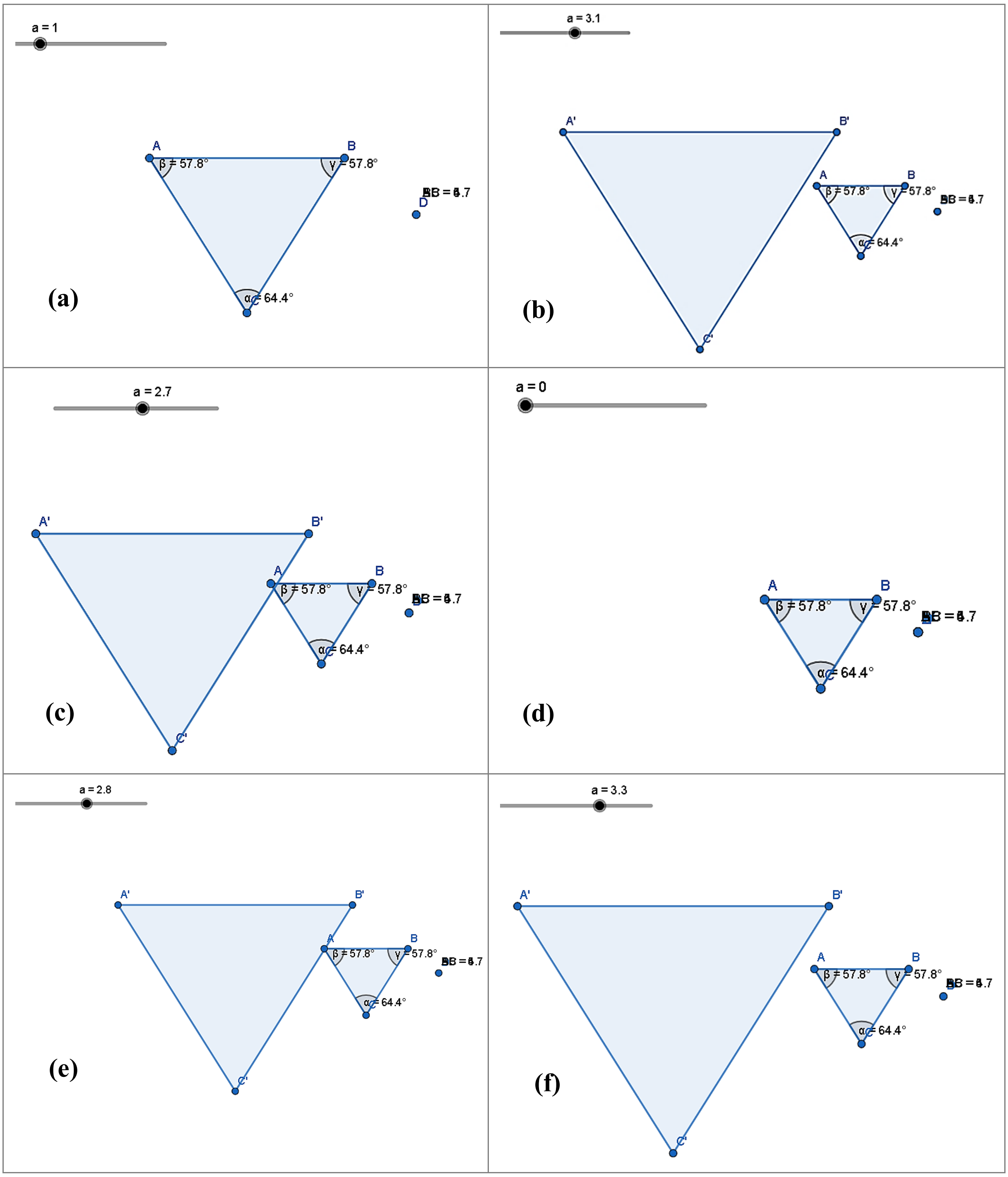
5.2 Episode 2: sliding control and the ratio of similitude.

In this second episode in the application of the second task in the didactic sequence, three (prospective) teachers used the sliding control to explore the ratio of similitude between two triangles. The aim of the task was to build any triangle, a sliding control, and a second triangle dependent on the first, through the function of "homothety" of GeoGebra. The script proposal was to direct the participants to explore the angular properties and the ratio between the length of the sides of the similar triangles.

The analysis of this episode aimed to understand the role of sliding control in the process of semiotic mediation. To do this, we sought the interpretation of the aim of the participant's dragging through the crossing of the written registers on the chat, both before and after the dragging movement of the sliding control.

Nicole and Maria, who participated in the previous task, were also involved in the task with sliding control. The team consisted only of undergraduate students, and their fictional names are: Júlio, Suzi, Nicole, and Maria. Júlio came in one hour earlier than planned and wrote down his initial remarks in the chat. Some of them were considered by the group during the proposed task. After the construction of the sliding control and depending triangles, the members of the group started to alter the value of the sliding control and, later, shared their remarks in the chat. Chart 4 presents the data from Replayer during the exploration of sliding control by Nicole.

Chart 4: Process of alteration of values of the sliding control by Nicole (19:44:05 – 19:44:45)



Source: Own elaboration, based on data from the *Replayer*

The values explored by Nicole belong to the interval between zero and five (Chart 4, figures a to d) and then concentrate on values higher than one (Chart 4, figures e to f). The same pattern in the variation of values repeated itself at the later moment, when Maria dragged first between intervals from one to five, and then altered to the value of zero and to values over five. Both students explored in a wider way the sliding control between the initial value and the last. Therefore, the remarks in the chat relate to more general aspects. Maria observed the existence of the relation of the movement of the sliding control with the enlarging and reduction of triangle A’B’C’, and Nicole conjectured about the distancing or approaching of the triangle to

D, the point where the homothety function from GeoGebra was applied, to the extent that it altered the values of the sliding control.

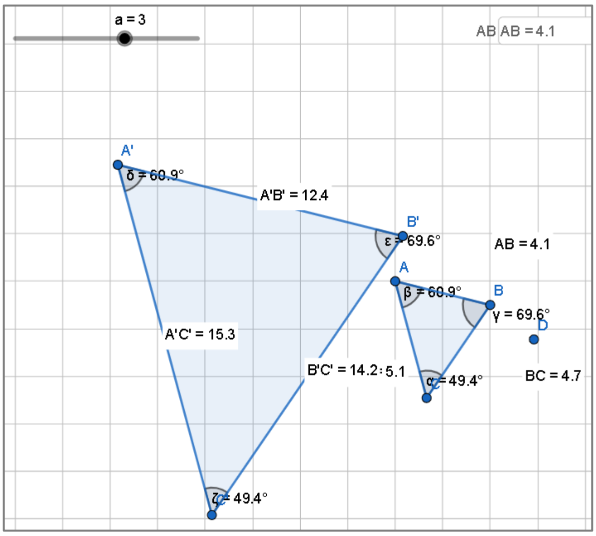
Maria and Nicole generated different signs, although they were using the same artifact, as they were sharing with the group their personal meanings, resulting from their own experience with the sliding control. Later the participants explored more specific values of the sliding control. They noted cases as a = 1 and a = 0, and they observed that the triangles are congruent and that in the second case the distance from the points in the triangle to point D is zero. In relation to the first moment, there was an advancement of the most generic aspects to the more specific ones, to the extent that the personal meanings began to converge.

The participants still had not interacted about the interval between one and zero, neither had they measured the length of the sides, only the internal angles. They used the random dragging to discover the regularities of the figure, and by the reaction of the environment they identified the corresponding angles and observed that they were equal. The members of the group measured the length of the sides of the triangles.

When we reached step 8 in the task script, we asked the participants: "Compare the length of the sides of triangle A'B'C' with the corresponding sides of triangle ABC. To do so, select the entrance field with icon a=1. In this field, calculate the ratio of segment AB by A'B'". The participants tried to insert the length ratio of the sides in the entrance box of the commands, but were not successful, because of a limitation of VMT which didn't make the insertion of formulas available.

The participants questioned the mediating teacher about a problem with the entrance box for the GeoGebra commands and were oriented to use the calculator in their computers to obtain the ratio between the corresponding sides. The final configuration in the Graphic zone at that instant in the task is shown in Figure 3.

Figure 2: Configuration of Graphic zone, the data of which were extracted to use with the calculator (146-167)



Source: *Replayer* Data

Chart 5 presents the segment of interaction among participants and mediator of the group-Author, when they were using their computer or smartphone calculators and sharing the results in the chat. Artifact signs and mathematic signs were identified. We are representing those signs by the symbols "<" and ">". The pivot sign, the sliding control <a>, is represented between the columns of Chart 5 in the text boxes.

Chart 5: Synthesis of the evolution from artifact to mathematic signs

Index	Participant	Posting in *chat*	Artifact sign	Mathematic sign
148	Author	The other groups also had a problem with this tool. You can use your computer calculator in case you cannot use this tool.	**<a>**	
149	Nicole	Because the ratio would be A'B'=**a***AB	<A’B’>, <a>	<A'B'=a*AB>
150	Nicole	And so on, for each side	<side>	
151	Maria	OK		
152	Maria	8) ratio AB/A'B'= 0,3307		<ratio>
153	Maria	Approximately that value		
154	Nicole	8) the ratio for **a**=3, A'B'/AB, is 3	<a=3>	<A'B'/AB, is 3>
155	Suzi	It’s AB/A'B'		<AB/A'B'=0,3307>
156	Maria	I did the ratio for **a**=3, but for AB/A'B'=0,3307 approximately		
157	Nicole	Doing AB/A'B' it also was 0,3306		<AB/A'B' also was 0,3306>
158	Maria	9) **a**=3, ratio BC/B'C'= 0,3309	<a=3>	<BC/B'C'=0,3309>
159	Nicole	BC/B'C'= 0,3309	<a=3>	<AC/A'C'=0,33333>
160	Maria	9) **a**=3, ratio AC/A'C'= 0,33333		
161	Nicole	AC/A'C'= 0,333...	**<a>**	<1/3>
162	Nicole	For me, the ratio of AC and A'C' is the most precise, because it would be 1/3 which is exactly 0,333...		
163	Author	does this result have anything to do with the value of "**a**"?	<a>	
164	Nicole	I think that **a** is the ratio	<a> **<a>**	<ratio>
165	Maria	Yes, altering the value of **a,** the triangle A'B'C' changes its size, therefore the ratio changes	<triangle A'B'C'>	
166	Nicole	That’s why it says that the ratio from ABC to A'B'C' is 1/3	**<a>**	<ratio from ABC to AB'C' é 1/3>

167	Nicole	Now you can see from the side AC=6, as **a**=4, then A'C'=24		

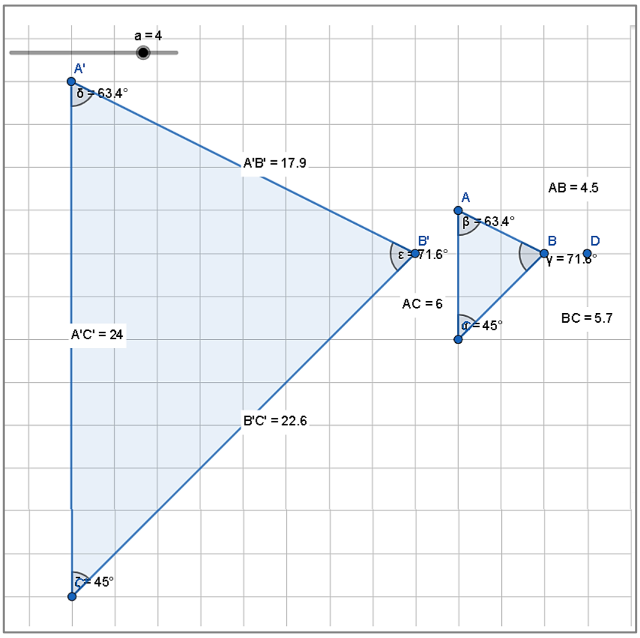
Source: Own elaboration based on data generated in VMT

Various artifact signs were generated by the participants, regarding the constructions manipulated in the graphic zone. In the same way, mathematic signs were generated with reference to the value obtained by the ratio of corresponding sides, represented by the number in decimal modality. The sign <a> of the sliding control was often referred to and played the role of connecting the two contexts to promote the passage from personal meanings to mathematic meanings represented by the respective ratios of the triangle sides.

Nicole realized that, performing the ratio of side AB to A’B’, the values were close to de decimal representation 0,333 or 1/3 in fractional form (Chart 5, index 162). As she noticed that the sliding control was set at a value of three, Nicole conjectured that the value of the inverted ratio of 1/3 was being represented by the value of the sliding control <a=3> (Chart 5, index 154). So, she proposed to her mates they calculate the ratio of A’B’ for AB.

We also registered that Nicole manipulated the sliding control again and, more intentionally this time, she modified the sliding control to the whole value <a=4> and she obtained the length AC = 6 e A’C = 24, as shown in Figure 4.

Figure 3: Configuration of the Graphic zone that comprises the descending process



Source: *Replayer data*

This is how Nicole concluded that A’C is equal to the product of <a> by the length of segment AC (Chart 5, indexes 149, 167). Therefore, Nicole used a process with a descending flow, from the theory to the construction, thus checking the validity of the property and that it was kept for other measures of the sides of the triangles. In this process, the calculator was used, with the aim to construct a conjecture about the ratio of the length of the sides of A’B’C’ by ABC, and the process of altering the values of the sliding control played the role to help in the production of conjectures and the validation of an observed property.

6 Conclusions

In this article we presented theoretical referents and empiric examples about prospective mathematics teachers interacting in VMTwG in activities concerning triangle similitude. The theoretical references we have approached can be useful to the (prospective) teachers in the creation of discursive strategies when promoting the continuity of interaction in a task and accompanying the phase of task resolution and the cognitive processes of their students. It is also useful to the comprehension of the process of semiotic mediation to evaluate the evolution of artifact signs and mathematic signs through a pivot sign.

In episode 1 we focused the analysis on the role of the checkered mesh of triangles. The checkered mesh worked as a pattern, making it possible to build congruent triangles in a direct way. Nevertheless, the non-dependency between triangles and the use of checkered mesh limited the exploration to some properties and relations of a more general aspect and limited to some types of triangles. We identified the flow of thought as ascending, from the construction to the theory, and the resolution of the task limited itself to the discovery phase.

In episode 2, from the measures of the sides and the calculus of the ratios with the calculator, the participants constructed the conjecture of the proportion of the corresponding sides. The use of the sliding control enabled us to explore approaches, inferences on measures and the notion of ratio as fraction, the analysis of global and particular cases (a = 1 and a = 0). The processes involved in the use of sliding control can assume a more powerful form, as they allow not only global observation, but also the checking. The types of flow of thought identified were ascending and descending, and encouraged exploration, building conjectures, and intentionally checking their suppositions.

Summing up, based on the theoretical framework of our work it was possible to have a notion of the semiotic mediation process of thought involved in the actions of dragging, either by free points or sliding control. We hope that this article can contribute to other mathematics teachers' exploration of the semiotic potential of geometric tasks with or without the use of VMTwG and that they can help their students in the learning of triangle similitude and other geometry subjects. We would like to stress the importance of VMTwG as an environment of interrelated spaces that enables a constant interactive process, where the students themselves explore, interpret, and check their thoughts.

We must warn of the fact that, as it is a robust system involving various functions, a good connection is required, so that all tasks may be carried out without obstacles. The VMT environment is in constant construction and relies on a technical support team who listens to suggestions to the problems reported by the users. Some suggestions were made through our own research, including the availability of the box to the access of VMTwG commands, following the problem that took place in episode 2. This was duly solved, and it is already possible to insert commands for the sides' ratios directly in the environment.

This research related to an MA program had its educational outcome[4]: *Sequência de atividades sobre semelhança de triângulos no ambiente Virtual Math Teams* (Sequence of activities on triangle similitude in Virtual Math Teams environment) where we present, besides the tasks mentioned above, other tasks, discussing the semiotic process involved in each task, and give suggestions to the teachers. We welcome the reader to get to know, develop and adapt the tasks with VMT among their students. As a closing remark, let us highlight that VMTwG is one of the scenarios that can take place in training, and we would not boast about it being the

[4] Available at: https://gepeticem.ufrrj.br/sequencia-didatica-sobre-semelhanca-de-triangulos-no-ambiente-virtual-math-teams, access April 24. 2022.

only one. On the contrary, let's rejoice in the fact that it is there as one more field for the exploration of mathematics concepts.

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