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Manipulative materials as mediation artifacts in problem solving

André Augusto Deodato
Universidade Federal de Ouro Preto
Ouro Preto, MG — Brasil
✉ andre.deodato@ufop.edu.br
0000-0003-4323-4010

Juliana Batista Faria
Universidade Federal de Minas Gerais
Belo Horizonte, MG — Brasil
✉ julianabatistafaria@gmail.com
0000-0003-1315-0094

Roselene Alves Amâncio
Universidade Federal de Minas Gerais
Belo Horizonte, MG — Brasil
✉ roseleneamancio@ufmg.br
0000-0001-9118-528X

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Abstract: Curricular documents encourage the use of manipulative materials, revealing an expectation that they will contribute to the improvement of the teaching of Mathematics in basic education. In this article, we intend to analyze some productions of twelve by 6th grade students who used in the second semester of 2018 manipulative materials to solve two problems. Based on a Historical-Cultural perspective of learning, we seek to understand the potential of manipulative materials beyond planning, focusing on how learners appropriate these objects. Grounded in the assumptions of qualitative research, we analyzed students' resolution strategies based on the reports of classes produced by undergraduate Mathematics students who were collaborators in the research. As the main results, we highlight that the appropriation of the manipulative material by the students led the resolution process and to the understanding of the possibility of replacing the manipulative material with a design. Thus, we infer that the use of such materials in addition to motivating is justified when they mediate problem solving.

Keywords: Artifacts. Manipulative Materials. Problem Solving. Elementary School. Mathematics Education.

Materiales manipulativos como artefactos de mediación en la resolución de problemas

Resumen: Los documentos curriculares fomentan el uso de materiales manipulativos con la expectativa de que contribuyan a mejorar la enseñanza de las Matemáticas en la educación básica. En este artículo pretendemos analizar algunas producciones de doce alumnos del 6º grado de la educación primaria que utilizaron, en el segundo semestre de 2018, materiales manipulativos para resolver dos problemas. Desde una perspectiva Histórico-Cultural, buscamos comprender el potencial de los materiales manipulativos centrándonos en las apropiaciones que los aprendices hicieron de estos objetos. Anclados en los supuestos de la investigación cualitativa, analizamos las estrategias de resolución de los estudiantes con base en los informes de clase producidos por los estudiantes universitarios que participan en el proyecto de investigación. Como resultado, destacamos que la apropiación de material manipulativo llevó a los estudiantes a utilizarlo como un artefacto de mediación y a desarrollar nuevas comprensiones sobre la función del dibujo. Así, inferimos que el uso de dichos materiales, además de motivar, se justifica cuando median la resolución de problemas.

Palabras clave: Artefactos. Materiales Manipulativos. Resolución de Problemas. Educación Primaria. Educación Matemática.

Materiais manipuláveis como artefatos de mediação na resolução de problemas

Resumo: Documentos curriculares incentivam o uso de materiais manipuláveis com a expectativa de que contribuam para melhorar o ensino de Matemática na educação básica. Neste artigo, pretendemos analisar algumas produções de doze estudantes do 6º ano do ensino fundamental que utilizaram, no segundo semestre de 2018, materiais manipuláveis na resolução de dois problemas. Baseando-nos em uma perspectiva Histórico-Cultural, buscamos compreender o potencial dos materiais manipuláveis ao focalizar as apropriações que os aprendizes fizeram desses objetos. Ancorados nos pressupostos das pesquisas qualitativas, analisamos estratégias de resolução dos estudantes com base em relatórios de aulas produzidos por licenciandos participantes do projeto de pesquisa. Como resultados, destacamos que a apropriação do material manipulável levou os estudantes a utilizarem-no como artefato de mediação bem como a desenvolverem novos entendimentos sobre o papel do desenho. Assim, inferimos que o uso de tais materiais, para além de motivar, justifica-se quando eles medeiam a resolução de problemas.

Palavras-chave: Artefatos. Materiais Manipuláveis. Resolução de Problemas. Ensino Fundamental. Educação Matemática.

1 Introduction

The official documents that guide curricula in Brazilian schools (Brasil, 1998, 2017) encourage the use of different teaching resources, showing an expectation that they will contribute to improving the teaching of Mathematics in basic education. This is an expectation that is often expressed in the speech of many teachers interested in making their classes 'more attractive' for students, when they face difficulties in getting good results on school tests. Work in the field of Mathematics Education that problematized the thoughtless use of these resources at the end of the 20th century (Fiorentini & Miorim, 1990) already showed that many of these teachers are not always clear on the reasons why these materials would be important for the teaching and learning process of mathematics.

In this article, we intend to shed some light on this discussion, describing and analyzing some productions by elementary school of twelve students who used a type of teaching resource — manipulable materials — in problem-solving. These students study at Centro Pedagógico — full-time elementary school — of the Universidade Federal de Minas Gerais and have participated, during the second semester of 2018, in classes in a *Differentiated Working Group* (DWG) titled "*DWG Desafios Matemáticos*". This DWG is one of the actions of the extension project called "Descobridores da Matemática", which involves research, teaching and extension activities, with the following objectives: to promote the learning of mathematics through problem solving; to analyze strategies used by children from the 3rd to the 6th year of elementary school to solve problems; to develop a problem bank (selected, adapted and unpublished) that can be used to improve the process of mathematics teaching and learning; to provide opportunities for the professional development of basic education teachers and mathematics undergraduate students. The project is carried out in different schools in the metropolitan region of Belo Horizonte, but, in the context of the Centro Pedagógico, it is articulated with the curricular matrix of this school as a time-space for teaching learning and scientific initiation (by UFMG undergraduates who work there), as well as for strengthening and diversifying the learning of elementary school students who study in a full-time school.

Within the scope of the research that gives origin to this article, it is the class reports produced by the undergraduates that provide the empirical material used in our analysis.

In the second half of 2018, data collection period, in the project participated ten mathematics undergraduate students who acted as monitors; five professors from the application school who guided them; and one professor from the Institute of Exact Sciences of the university, who coordinated it. The teacher training work was developed through a weekly 80-minute meeting of each advising teacher with a pair of monitors in order to: plan the classes that would be conducted with a group of up to twelve children; talk about the classes taught; read and indicate, in the reports produced about the classes, which aspects could be improved in order to make them, pedagogically, denser. This orientation process was sustained beyond the weekly meetings, in email and *WhatsApp* message exchanges, and in some instances of joint production of materials to be used in classes.

One of the actions of the project, it is worth mentioning, is to carry out research movements that focus our view on certain aspects of the mathematics teaching-learning process documented by the undergraduates in their reports. Thus, in order to carry out this action, for the scope of this article, we turn our focus to two problems that were solved by twelve 6th grade students who participated in the classes of the *DWG Desafios Matemáticos*: the “tile problem” and the “six rectangles problem”. From the pedagogical point of view, these problems were selected because they allowed the elaboration of manipulable materials that could represent the geometric figures in question, since, in previous classes, several students had demonstrated difficulty in solving problems that required imagining geometric figures in positions different from those presented in the statements. From the point of view of the research, we selected these problems due to the quality of the reports prepared by the two undergraduates involved, which included photographs that showed, at least in part, how the students of the aforementioned DWG appropriated these materials to solve them. In our analysis, by describing and analyzing the manipulable materials used, we will seek to understand them as mediation artifacts in problem solving, based on a Historical-Cultural perspective of learning.

Thus, this introduction sought to briefly characterize the context of a qualitative research, prepared by three teachers (two women and one man), working in basic education, concomitantly with the processes of guidance of mathematics undergraduate students taking the *DWG Desafios Matemáticos*. It is from this meeting of research and training that the proposition of this research arises. Now the justification and problematization of the research will be described, referenced in studies belonging to the fields of mathematics education and historical-cultural perspective.

2 Theoretical and methodological background

Previously mentioned, the proposition of the *DWG Desafios Matemáticos* happens in a full-time school, articulating the teachers training to the training of children who participate in it. In this context, in addition to the specific objectives of the already mentioned Descobridores da Matemática project, some broader questions about the teaching and learning processes that occur within full-time schools are guiding our research and training work: how is mathematics taught and learned in these schools? How does the extension of school time contribute (or not) to mathematics learning? How are the mathematical practices in these schools characterized?

According to Deodato (2017), there is a strong demand in Brazil to record, in the literature, the mathematical practices that occurred inside full-time schools. Our work, in this article, seeks to contribute to this record, making a very specific cut, aimed at answering the following question: how, within the scope of the discipline *DWG Desafios Matemáticos*, students make use of manipulable material to solve problems? The DWG is a curricular component used by the school, field of research, to operationalize its full time. We consider that, by giving visibility to the productions of 6th grade students, we meet, in a way, the need enunciated by Deodato (2017).

Another aspect that situates our investigation in a broader scenario is the boundaries it shares with the literature on problem solving; with works that are dedicated to understanding the function of mediation artifacts used by students in the search for the school mathematical object; and with authors who make use of the historical-cultural tradition, in particular the Activity Theory, as a lens to analyze the students' productions.

In this sense, David, Tomaz and Ferreira (2014), for example, use the same theoretical framework during a mathematics class to discuss the agency power of a visual artifact, identified by them, that is, the 'little shower'. Especially, in the context of problem solving, Deodato, Faria and Amâncio (2022) have the Theory of Activity to ponder the productions of records made by 5th grade students. Still on this framework that we use, Takinaga and Manrique (2022), when seeking theories capable of offering "necessary support for research involving the teaching and learning of academic content, in the inclusive school context" (p.76), conclude that, in this case, the Activity Theory is one of the theories that hasn't been sufficiently explored so far.

The choice we made to discuss manipulable materials, interpreting them as mediation artifacts in problem solving, demands a dialogue with the literature in order to outline our understanding of some concepts that provide the structure of our analysis. To this end, this section is organized in three parts: in the first, we present some reflections regarding the notion of manipulable materials; in the second, we characterize two concepts that are fundamental for our analysis, namely: artifacts and appropriation; finally, we discuss our methodological choices.

2.1 The use of manipulable materials

What we know as the science of Mathematics deals with many abstractions, that is, with mental constructions that were developed by different peoples throughout history, both to solve everyday problems and for the pleasure of developing new ideas and making discoveries. These abstractions developed through relationships that human beings built with themselves, with others and with the world, largely mediated by objects used in their interaction with nature, society and culture. As Caraça (2010) states, mathematical concepts arise "once problems of fundamental, practical or theoretical interest are posed" (p. 118). The intellectual and conceptual instruments that allowed human beings to study the mathematical laws we know today developed gradually throughout history: "There was a slow gestation, in which need and instrument interacted, helping and clarifying each other mutually" (Caraça, 2010, p. 118).

In the classroom environment, students deal with one of the faces of this knowledge that was historically produced by humanity — "School Mathematics" (Moreira & David, 2005) — , learning and producing mathematics in the activities they carry out with their peers and with teachers. In many of these activities, manipulable materials are used, so called because they are "objects or things that the student is able to feel, touch, manipulate and move. They can be real objects that have application in everyday life or they can be objects that are used to represent an idea" (Reys 1971 *apud* Matos e Serrazina, 1996, p. 193). The use of these materials occurs in different ways in the daily lives of schools, depending on how teachers understand the insertion of this resource in their classes.

In the main two Brazilian curriculum documents (Brasil, 1998, 2017), we find recommendations that contemplate the use of manipulable materials in teaching. One of the guiding principles of the Parâmetros Curriculares Nacionais states that *"*didactic resources such as books, videos, television, radio, calculators, computers and *other materials* play an important role in the teaching and learning process" (Brasil, 1998, emphasis added). In the Base Nacional Comum Curricular (BNCC), it is stated that manipulable materials can "arouse interest and represent a significant context for learning and teaching Mathematics (Brasil, 2017, p. 296). Both documents warn teachers about the need to clarify their pedagogical intentions regarding the use of these materials and about the excessive expectations that teachers may have in relation to the effects these materials have on learning. The BNCC, for example, links the use of manipulable materials to situations that provide "reflection" and are intended to help students "systematize and formalize" the mathematical concepts involved.

In our work as professors and researchers, we consider that manipulative materials are teaching resources that are at the service of student learning. These materials can be used both in the sense of giving materiality to abstract ideas, helping students to understand mathematical concepts and procedures, and in the sense of being instruments for solving problems and for "mathematical research in the classroom" (Ponte, Brocardo & Oliveira, 2003). They can be brought to the classroom by the teacher or produced by the students themselves, with the teacher's guidance. Assuming the cultural-historical perspective, we consider that manipulable materials can constitute artifacts thought by teachers to mediate the student subjects' relationship with the desired (mathematical) learning object.

Several authors in the field of research in Mathematics Education have highlighted the potential of using manipulative materials for learning Mathematics and have also pointed out that the way in which the teacher proposes their use is essential in this process. However, there is still a conceptual imprecision that prevents more assertive statements from being made about them (Rodrigues, 2011). In any case, we consider it important to bring to this study some of the considerations made in the field.

Lorenzato (2006) observes that manipulable material is an auxiliary means of teaching, a methodological alternative available to teachers and students, and is not a guarantee of good teaching or meaningful learning. This aspect that questions the effectiveness of the manipulable material is also discussed by Fiorentini and Miorim (1990). According to these authors, many Mathematics teachers use manipulative games and materials only when they cannot obtain the expected results in their classes. In this situation, these materials are used only with the justification of making classes more attractive to students, without reflecting on the contributions and limitations of these resources for learning. This thoughtless use of materials can even have a negative effect on learning, since, paradoxically, the materials can become more abstract for students than the mathematical concept in question. The same authors emphasize that no material is valid in itself; however, they can be used in Mathematics classes in order to contribute to "a meaningful learning in which the student participates by reasoning, understanding, reworking the knowledge historically produced and thus overcoming their naive, fragmented and partial view of reality" (Fiorentini & Miorim, 1990, p. 4).

Another important aspect to be considered in the use of manipulable materials in the classroom concerns "who" will use them. As Matos and Serrazina (1996, p. 197) state: "It is different for a material to be used as a communication tool by the teacher who explains by showing objects that he alone manipulates, or for the students to manipulate them, interpreting their characteristics, solving problems with their help and formulating new problems". In our understanding, manipulative materials are teaching resources that the teacher can use to communicate mathematical ideas, but that need to be incorporated into the activities that students carry out, so they can use them in problem solving and/or in mathematical investigations experienced by them.

Considering the aspects mentioned here, our study seeks to contribute to the discussion about manipulable materials from the point of view of appropriations that students make to solve problems. We will deal more specifically with solving plane geometry problems. According to Pais (2000), the construction of geometric concepts is complex, involving intuitive, experimental and theoretical aspects. The author identifies two equally limited positions in the teaching of geometry: one that conceives geometric notions as purely rational abstract entities and another that restricts the teaching of geometry to experimental activities, through the simple manipulation of material objects and drawings. The author argues that the teacher should seek a point of balance between rationalism and empiricism, so that experimentation activities are associated with some rational assumption and that deductive arguments are accompanied by some experimental dimension. It is in this sense that the author sees, in the act of handling materials, a potential to help students in learning geometry.

2.2 Mediation artifacts in the Historical-Cultural perspective

In our understanding, the potential of manipulable materials is not in the materials themselves, but in the way people appropriate these materials, making them mediation artifacts between the subject and the mathematical object in a certain activity. The notions of "activity", "appropriation", "artifact", "subject" and "object" that we mention in this statement are anchored in the Historical-Cultural perspective, referenced, especially, in the contributions of Vigotski, Leontiev and Engeström.

Among these contributions, we highlight, above all, the notion of human activity, which allows the delimitation of a unit of analysis that is not polarized either in the object of desire or in the human, but in the relationship (mediated by artifacts) of the human subject with his or her object of desire.

Deodato (2017), drawing on this understanding of the activity, resorts to Engeström and Sannino (2010) to highlight that the subjects' points of view are chosen as the perspective of analysis. He also adds that artifacts are instruments (or signs) that mediate the subject's relationship with the object of the activity. It also states that the object represents the "problem space" towards which the investigated activity is directed. Taking these understandings as a perspective, in this article, we consider that 6th grade students are subjects of an activity whose object is problem solving. Furthermore, in this activity, the appropriation of a certain artifact (cardboard) by the students reveals, as will be seen below, ways in which the manipulable materials serve the development of mathematical learning processes.

According to Lopes and Marco (2015), the historical-cultural framework has been used recurrently to produce analyses in investigations related to Mathematics Education. In an investigation carried out by these authors it was found that this approximation between Mathematics Education and the historical-cultural perspective is perceived in the organization of various research groups registered in CNPq: of the 348 groups anchored in the aforementioned framework, 21 had professors or specific research lines linked to Mathematics Education. The authors, to elucidate this situation, assert:

> The concern with teaching and learning mathematics in school education, especially in the processes that are consolidated in public schools, has led many researchers in the field of Mathematics Education to seek in Historical-Cultural Psychology the understanding that teaching is of great importance in development of the individual and that the relationship between the teaching activity — of the teacher — and the

> learning activity — of the student, is centered on the way the educator organizes teaching (Lopes & Marco, 2015, p. 458).

It is precisely this concern with the organization of teaching and learning of Mathematics, within the public school, connected to the understanding of the role of teaching for the development of the human subject, that explains the theoretical-methodological basis that supports the assumptions of this article.

Therefore, drawing on this foundation, in the scope of studies more directed to Mathematics Education, our reference for thinking about the "appropriation" of manipulable materials is in the work of Smolka (2000, p.28), according to which "appropriation refers to ways to make proper, to possess; also, make adequate, relevant, to the socially established values and norms". This concept helps us identify and understand the uses that students make of these materials to think and do mathematics and how these students participate, with these materials, in the interactions that take place in the classroom. In this appropriation process, we can see manipulative materials as "artifacts", in the understanding we seek in Cole (1996, *apud* Costa, 2016):

> [...] artifact is an aspect of the material world that has been modified throughout its history by incorporation into goal-directed human action. Due to the changes introduced in the process of their creation and use, artifacts are simultaneously ideal (conceptual) and material (Cole, 1996, p. 117 *apud* Costa, 2016, p. 55).

In this article, manipulable materials are, therefore, artifacts that mediate the relationships between subjects and mathematical objects. Our understanding of "subject" and "object" is in line with that produced by Engeström and Sannino (2010), according to which: (i) the subjects are the people whose point of view is adopted as an analysis perspective; (ii) the object is the 'raw material' or the 'problem space' towards which the investigated activity is directed. Using this understanding, we understand that artifacts can be appropriated by subjects when they, in *Human Activity*, pursue their object of desire.

2.3 Methodological procedures

Having, therefore, built understandings for the concepts of manipulable materials and artifacts, below we characterize our methodological procedures.

This is qualitative research and makes use of documentary analysis (Lüdke & André, 1986) as a way to produce knowledge. Lüdke and André (1986) refer to documents in a broad way, which includes "laws and regulations, standards, opinions, letters, memos, personal journals, autobiographies, newspapers, magazines, speeches" (p. 38). In the case of our study, the documents assessed were the reports produced by undergraduates in partnership with their advisors, within the scope of the Descobridores da Matemática project.

Three stages constituted our methodological path. Initially, we conducted a search in the Descobridores da Matemática project report collection. These reports, we must indicate, highlighted the solutions that 6th grade students produced to the problems worked out during the *DWG Desafios Matemáticos*, from the perspective of the undergraduates, guided by UFMG professors. Thus, they intended to describe the taught lesson, with emphasis on different strategies for solving the proposed problems.

Then, we selected those documents in which we found reports on the resolution of problems in which elementary school students made use of manipulable materials. We must definitely highlight that the level of detail of the reports also contributed to this selection, since they did not have a rigid format, that is, the reports were produced based on each undergraduate student's personal logging strategies. Thus, it is worth informing that our empirical material consisted of reports on the resolution of the tiles and six rectangles problems, both characterized in detail later in this article.

The final phase of the course involved an interpretation of the uses of manipulable materials during the problem resolution, based mainly on the concept of artifact interpreted through the Historical-Cultural perspective. We emphasize that this concept enabled the production of considerations about the appropriation by 6th grade students of manipulable materials as mediation artifacts in the resolution of two problems. Finally, we should repeat that the analyses produced here were made *a posteriori*, based on documents whose production took place in the context of the undergraduates training, limiting our research work to the problem solving possibilities that were reported by them.

3 Problem solving in the context of the Descobridores da Matemática Project

As already mentioned, one of the objectives of the Descobridores da Matemática Project is to contribute to the improvement in the learning of students from the 3rd to the 6th grade of elementary school, through the resolution of problems that are selected, adapted and/or elaborated in "orientation meetings" among undergrad Mathematics students from and professors from the Centro Pedagógico at UFMG. In these meetings, we analyze with future teachers some problems already available in the literature and make the adaptations deemed adequate for the classes of *DWG Desafios Matemáticos*. We also design new problems and produce manipulable materials that can be used by students during classes. In addition, we talked about the students' productions and instructed the undergraduates to record their observations in reports about the classes given. The records are made after the completion of each class. These are small accounts on how the classes took place, containing some pictures of the classroom and the students' productions. Not all productions are photographed, and the richness of the interactions that take place in the classroom is not always present in the way of reporting the experiences lived by the undergraduates. It is part of the orientation process to collaborate so that these reports grow in narrative and pedagogical density, with some of our interventions as we read and talk about what they write.

Our concept of "problem" is similar to that presented by Alevatto (2005, p. 41), who considers that "a question is a problem if the student does not yet know the means necessary to solve it, but is interested in solving it". Thus, we seek to propose tasks in the *DWG Desafios Matemáticos* that have the potential to arouse the interest of students and that do not have the possibility of an immediate answer, requiring them to develop strategies to solve the presented problem-situations.

In this section, we will present and discuss the resolution of two problems that demand visualization skills: the "tile problem" and the "six rectangles problem". Such an ability, according to Van de Walle (2009), can be considered the ability to create mental images of shapes and to imagine them in different positions. According to the author, tasks that contribute to the development of visualization skills should require students to "think about a shape mentally, manipulate or transform a shape mentally, or represent a way in which it is visually perceived" (Van de Walle, 2009, p. 474).

We had verified, throughout the experience in the DWG, that students had more difficulties in solving problems that demanded the need for geometric visualization. Based on this perception, we proposed other problems that required students to be able to imagine figures in different positions. In the search for a solution, students were able to explore manipulative materials that represented the situations proposed by the problems. For, according to Pais (1996), the formation of mental images is a consequence of the experience with objects and drawings.

3.1 The tile problem

We started one of the orientation meetings of the undergrads with the presentation of the "tile problem". It was taken from the question bank of the Olimpíadas Brasileiras de Matemática das Escolas Públicas (Figure 1).

Figure 1: Original problem – Pedro's Tiles

14 | Pedro's Tiles

Pedro is a mason. He has a large number of tiles in three shapes, as show below:

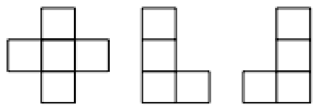


Figure 14.1

The smallest side of each tile measures 10 cm. He wants to completely cover a kitchen counter without cutting any tiles.

a) Show a way in which he can reach his goal if the counter is a 60 x 50 cm rectangle.

b) Show a way in which he can reach his goal if the counter is a 60 x 60 cm square.

Source: http://www.obmep.org.br/bq/bq2011.pdf

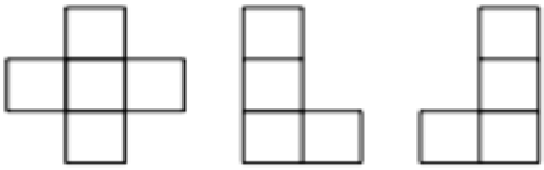
After pondering this problem with the undergraduates, we considered that the language of the statement was clear and adequate for 6th grade students; we also considered that, for such students, it would be necessary to think about mediation artifacts — in addition to the sketch — to help them develop solution strategies. Therefore, we understood that an alternative would be the construction of pieces with manipulable material to represent the tiles.

In our view, for such pieces to be appropriated as mediation artifacts between the subjects (students) and the mathematical object (problem to be solved), it would be necessary for them to have the dimensions indicated in the drawing; it would also be necessary for them to occupy an area that could be outlined in the classroom, at the students' desk. This understanding led us to make small adjustments to the original problem, as can be seen in Figure 2.

Figure 2: Adapted problem – Pedro's Tiles

Pedro's Tiles

Pedro is a mason. He has a large number of tiles in three shapes, as show below:



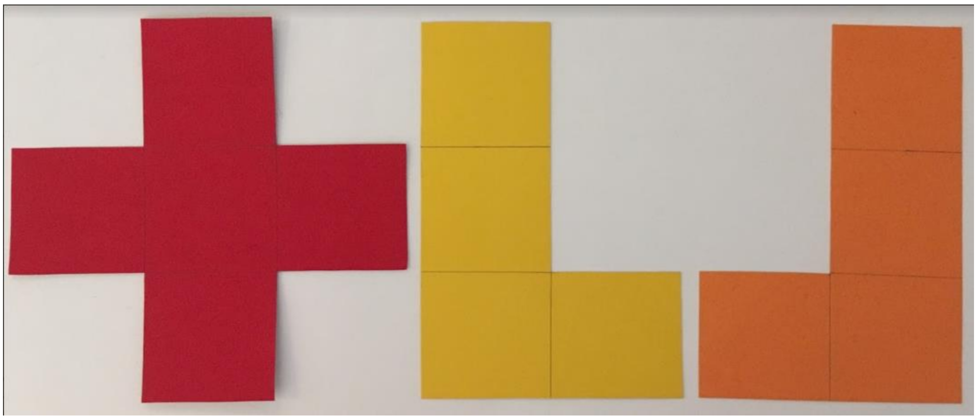
The smallest side of each tile measures 5 cm. He wants to completely cover a kitchen counter without cutting any tiles.

a) Show a way in which he can reach his goal if the counter is a 30 x 25 cm rectangle.
b) Show a way in which he can reach his goal if the counter is a 30 x 30 cm square.

Source: Descobridores da Matemática Project Collection

It should be clarified that we chose "paperboard" as the raw material to make the pieces that would represent the tiles (Figure 03) for two main reasons: i) the material was easy to handle both for teachers (in assembly) and for students (in use); ii) this material was available in the school's warehouse. Other materials could also be used, such as EVA and cardboard. The production of the records was started at the orientation meeting and enough material was made available for the two monitors to finish making them later.

Figure 3: Parts made of paperboard

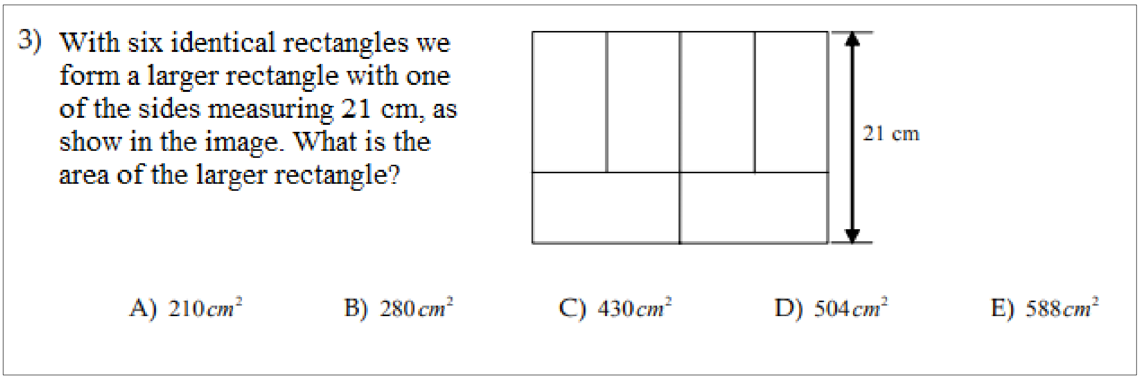


Source: Descobridores da Matemática Project Collection

3.2 The six rectangles problem

The "six rectangles problem" was taken from the question bank of the Olimpíadas Brasileiras de Matemática das Escolas Públicas (Figure 04).

Figure 4: Original problem – Six rectangles


3) With six identical rectangles we form a larger rectangle with one of the sides measuring 21 cm, as show in the image. What is the area of the larger rectangle?

A) $210 cm^2$ B) $280 cm^2$ C) $430 cm^2$ D) $504 cm^2$ E) $588 cm^2$

Source: http://www.obmep.org.br/bq/bq2010.pdf

As in the "tile problem", we consider that the text of the "six rectangles problem" did not present inadequacies in content or language. To manufacture the manipulable material, we maintained the criterion of considering the dimensions indicated in the problem figure and again used "paperboard" to make the smaller rectangles (Figure 05).

Figure 5: Parts made paperboard

Source: Descobridores da Matemática Project Collection

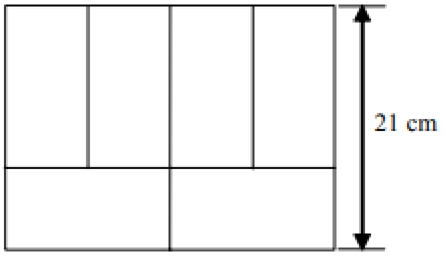
As we wanted, pedagogically, to favor the use of the manipulable material by the students, we chose to make a first adaptation in the text of the original problem We added the following observation to the problem statement: "Attention: Measuring the side of the rectangles is not allowed". When making this statement, we imagined that students would interpret it as an impediment to the use of conventional measuring instruments (such as a ruler, for example), which in fact happened. A second adaptation was made with the intention of shifting the focus of the problem from areas to perimeters. Finally, to encourage and allow students to come up with different solutions, we chose to turn a closed problem into an open problem. All these changes unfolded into a new problem, as shown in Figure 06.

Figure 6: Adapted problem – Six rectangles

Six rectangles

With six identical rectangles we form a larger rectangle, with one of the sides measuring 21 cm, as in the figure below. What is the perimeter of the largest rectangle, in cm?

Attention: Measuring the side of the rectangles is not allowed.



Source: Descobridores da Matemática Project Collection

In the same way as it happened with the production of materials for the "tile problem", again the production of paperboard pieces was started in the orientation meeting and finished by the monitors at another time.

4 Analysis of problems solved by 6th grade students

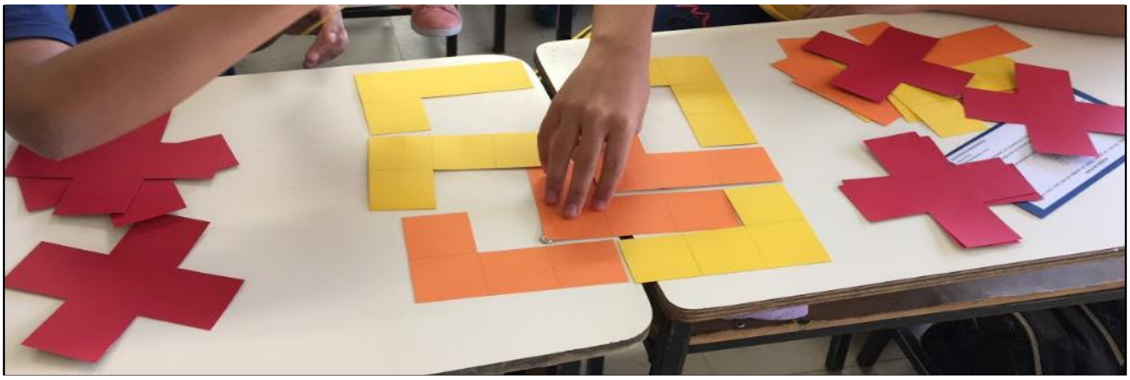
Once the two problems and the manipulative material used by the students, we analyzed the appropriations that the twelve 6th grade students taking the *DWG Desafios Matemáticos* made of this material as to develop their resolution strategies. To solve these two problems, students were organized into four three-student teams.

From a general point of view, it is possible to state that, in both problems, the 6th grade students had to imagine the geometric shapes of the tiles and rectangles in different positions and, to favor this process, they used the materials that were available to represent the situation of the problem. In other words, the students handled the pieces of cardboard, changing their positions, rotating the pieces, either in the representation of the tile (in the kitchen tiling) or in the representation of small rectangles (in the assembly of the larger rectangle), until they found a solution. In both problems, we could see that the students arrived at the solution through different strategies.

Specifically in relation to the first problem, we believe that it is important to highlight the role of manipulative material to help students express queries during the solution. While using the pieces to simulate tiling, one of the groups of students realized that it was possible to 'occupy the space' leaving empty spaces between the pieces and another group used all the pieces to 'occupy the space' promoting overlaps. Given this, we express the idea of "occupying space" in terms of a situation in the world of work: 'when a mason has to tile a surface, how does he do it?' It is not common to overlap tiles, nor is leaving empty spaces. Therefore, an opportunity was created for students to experience, through material handling, the visualization of solutions that are not validated as correct in the logic of solving this type of problem.

It is possible to highlight particularities in the way each group of students appropriated the materials. Regarding the "tile problem", one of the groups of students used the manipulative material to initially assemble 'the borders of the kitchen countertop'; that done, they had to think about the proper position of the other tiles to finish the 'internal' filling of the space (Figure 07). In the process, they rotated the pieces, trying different layouts, until they found the solution to the problem.

Figure 7: Parts delimiting the contour

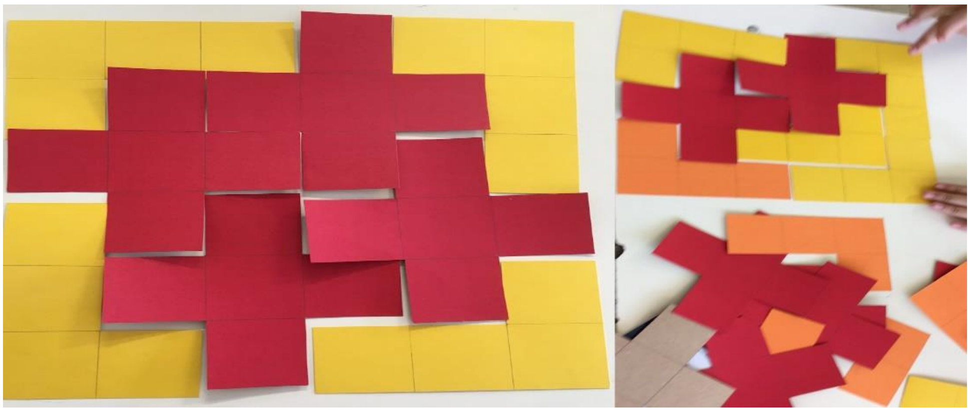


Source: Descobridores da Matemática Project Collection

This strategy, in our view, shows that the students invented their own use for the material with the intention of solving the problem, that is, they *appropriated* (Smolka, 2000) the manipulable material and used it as a *mediation artifact* (COLE, 1996), to face the difficult situation in which they found themselves.

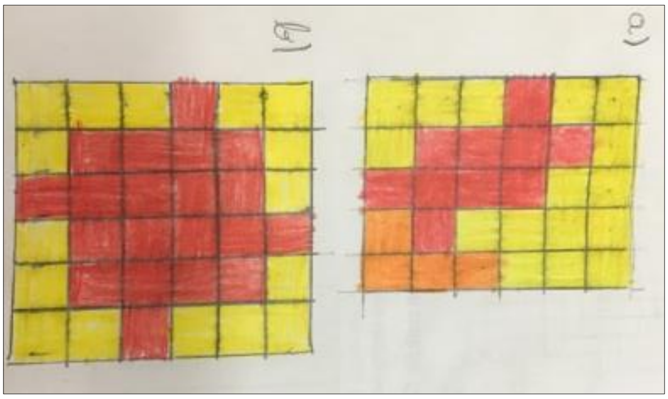
From the solution found by the students to this problem, illustrated in figure 08, they were provoked to reflect on the demand for the original problem, that is, how to occupy the space aided only by drawing. After some resistance and statements such as "it's impossible", with the mediation of the undergraduates who developed this task with them, the 6th grade students managed to build a representation through drawing (figure 09) that led them to conclude that, even if a possible lack of manipulable material made the process harder, it did not prevent them from reaching the same solutions. That is, they ended up realizing that, on one hand, the manipulable material has no value in itself, as indicated by Fiorentini and Miorim (1990), but that on the other hand, it can be appropriated in order to constitute itself as an *Activity* mediation artifact (Engeström & Sannino, 2010), in which they are the subjects who seek a certain object (mathematical).

Figure 8: Solutions found by the team



Source: Descobridores da Matemática Project Collection

Figure 9: Representation of the Solution through Drawing, performed by a student



Source: Descobridores da Matemática Project Collection

All teams were able to solve the problem, including representing the solution through drawings. It is important to highlight that this is a one-solution problem and, therefore, all teams produced drawings similar to Figure 9.

In relation to the "problem of the six rectangles", the process of appropriating the manipulable material as a mediation artifact demanded greater intermediation from the undergraduates who accompanied the students at the time of its resolution. After reading the problem and exploring the material, some 6th grade students began to say things such as: "*there is a lack of information about the sides of the small rectangle*", "*this problem has no solution*", "*this material does not help to solve the problem*", among others.

As they experimented with the material and with the assertive provocations of the undergraduates, two main strategies were produced by the students to solve the problem. To characterize them, we will use the following notation: lp (small side of the small rectangle), Lp (larger side of the small rectangle), lG (small side of the large rectangle), and LG (larger side of the large rectangle).

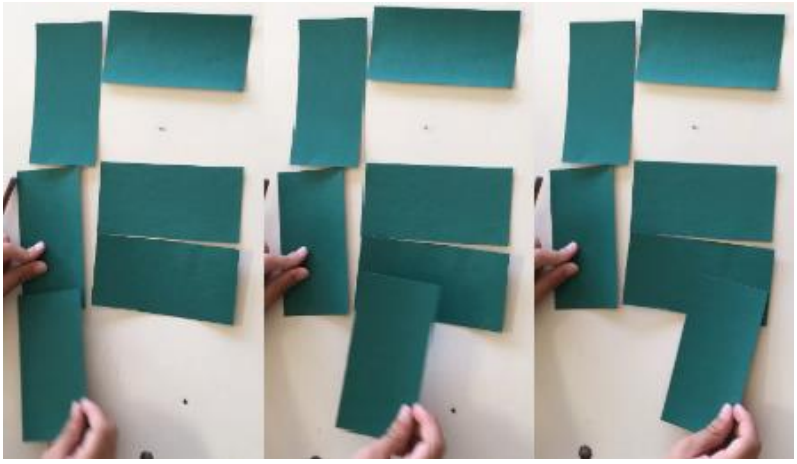
A first group of students, as they handled the material, began to attribute meaning to it, "making it their own" (Smolka, 2000). In this process, they noticed that the lp measurement could be used as a length unit of measurement (figure 10). From this discovery, they realized that the measure of Lp was equal to twice the measure of lp, or simply Lp = 2lp. From there, without much difficulty, they noticed that: lG = 3lp and LG = 4lp. During this stage, they returned to the drawing in the statement and also, with some ease, found the value of lp (7 cm), by dividing 21 (measurement of lG reported in the problem) by 3 (relation between lG and lp perceived by students). Hence, they concluded that the perimeter was equal to the sum: LG + LG + lG + lG = 4lp + 4lp + 3lp+3lp = 14lp = 14 · 7 = 98 cm.

Figure 10: Comparison between lp and Lp

Source: Descobridores da Matemática Project Collection

A second group of students found the solution in another way. The students initially constructed the larger rectangle and perceived, by handling, that *lG = 3lp* (figure 11). Once they reached this conclusion, they returned to drawing the problem statement and, like the other group, had no difficulty in concluding that lp = 7cm. Without further complications, they also noticed that LG = 4lp. After that, to calculate the perimeter they equated *2LG + 2lG = 8lp + 6lp = 14lp = 14. 7 = 98 cm.*

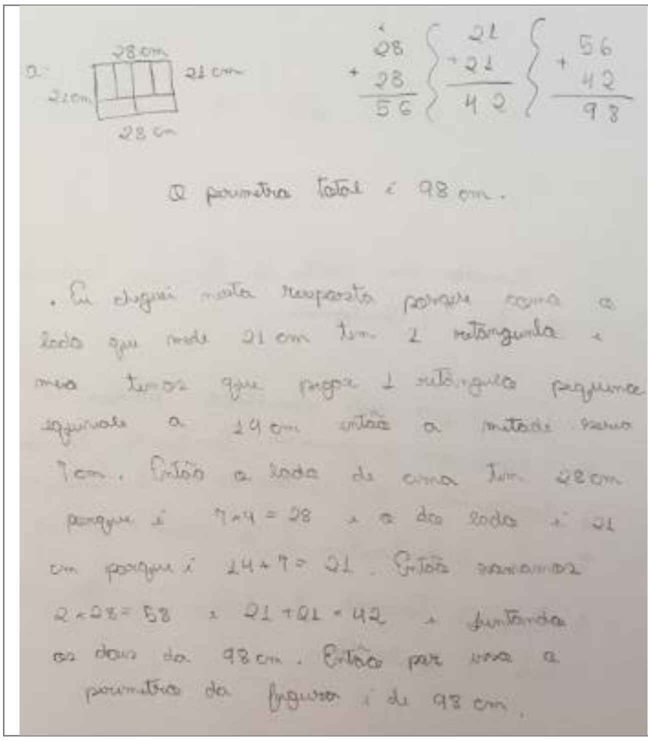
Figure 11: Comparing lp with lG



Source: Descobridores da Matemática Project Collection

Students were also encouraged to explain their thought problem to solve the problem, since, as stated by Pais (2000), it is important that manipulation is associated with an intellectual activity so that the student can establish a relationship between practice and theory. Below, we present one student's explanation about the solution to the challenge, with the transcript on the side to facilitate reading (Figure 12). This was the explanation highlighted as the most complete by the undergraduates.

Figure 12: Student's explanation of the solution to the "six rectangles problem"



The total perimeter is 98 cm.

I came up with this answer because as the side that measures 21 cm has 1 rectangle and a half, so we have to take 1 small rectangle equals 14 cm so half would be 7 cm. So, the top side is 28 cm because it's 7 x 4 = 28 and the side is 21 cm because it's 14 + 7 = 21. So, we add 2 x 28 = 56 and 21 + 21 = 42 and putting both together gives 98 cm. So that's why the perimeter of the figure is 98 cm.

Source: Descobridores da Matemática Project Collection

Thus, in both problems, students manipulated and rotated the pieces. To visualize them in different positions, they made attempts, so that the material was used to favor mathematical reasoning, with the active participation of the students, as warned by Fiorentini and Miorim (1990). Therefore, we understand that 6th grade students appropriated the manipulable material and found their own solutions, validated within the logic of School Mathematics (Moreira & David, 2005), for the proposed problems.

5 Final Considerations

In this article, we presented and discussed some strategies used by 6th grade students during the resolution of the problems shared in this study, seeking to analyze the appropriations that these students made of certain manipulable materials, available in the classroom, in order to help them develop their own strategies for solving such problems.

To discuss what was proposed, initially, we carried out a literature review on the use of manipulative materials in the context of Mathematics Education. After that, we presented our conceptual framework, emphasizing some notions linked to the Historical-Cultural perspective and a certain view on problems and geometrical visualization. Next, we presented the empirical material that we reflect on to produce our analysis.

We consider pertinent to highlight three aspects from this process. One of them is that, from the perspective of problem solving that we assumed, more emphasis was placed on the processes that the 'discovering students' developed than on the solutions they found and which were eventually considered correct in mathematical terms. Being a "discoverer" of Mathematics in this project is something concerning the subject who learns, related to the actions of creating, inventing and developing their own problem-solving strategies.

In the same direction, it is worth clarifying that we did not develop a work whose main logic was to 'train' students to solve a specific type of problem. While we were aware that they could become familiar with the 'grammar' of the Olympiad problems and we considered this a positive element, our intention was broader. We intended to develop work with basic education students, through which they could mobilize mathematical knowledge when creating strategies to solve problems. This claim is evident, for example, in the effort we make to adapt the statements of the problems. In the “tile problem”, this adaptation was more subtle and involved the choice of numbers that would make the use of manipulable material more significant. In the “six rectangles problem”, this adaptation led to a change in focus – from area to perimeter – that made the exploration of the problem more suitable for the group of students we were working with.

The second aspect that we underline is more specifically related to the use of manipulative materials. Although we recognize benefits in their use, we do not make a naive defense of the use of this type of material. On the one hand, we agree with Fiorentini and Miorim (1990) in the sense that the teacher should not submit their theoretical-methodological conceptions to some type of material, but, on the contrary, must submit the material to their theoretical-methodological assumptions. On the other hand, we think that manipulable materials must be justified not only because they can motivate students in Mathematics classes, but mainly because they can show themselves as mediation artifacts between the student subject and the mathematical object pursued.

We conclude this article by pointing out a third aspect. We realized that students not only appropriated manipulable materials in the search for a solution to a problem, but also became independent of them in this process. This statement is supported by what was observed during the “tile problem”. In it, the students used the pieces of cardboard to solve the problem, but also went beyond, when they presented evidence that they understood that the design of the statement was sufficient to solve the problem.

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References

Allevato, N. S. G. (2005). *Associando o computador à resolução de problemas fechados: análise de uma experiência.* 2005. 370f. Tese (Doutorado em Educação Matemática). Universidade Estadual Paulista. Rio Claro. São Paulo, SP.

Brasil. (1998). Ministério da Educação. *Parâmetros Curriculares Nacionais: Terceiro e quarto ciclos do Ensino Fundamental – Matemática*. Brasília, DF: MEC/SEF.

Brasil. (2017) Ministério da Educação. *Base Nacional Comum Curricular: Educação Infantil e Ensino Fundamental.* Brasília, DF: MEC/SEF. http://basenacionalcomum.mec.gov.br

Campos, C. A; Souza, M. R. A. & Silva, D. J. (2020). Dicionário Científico Ilustrado. *Interfaces-Revista de Extensão da UFMG*, *8*(1), 133-143.

Caraça, B. J. (2010). *Conceitos fundamentais de matemática* (7a. ed.). Lisboa: Gradiva.

Cole, M. (1996). *Cultural psychology: a once and future discipline*. Harvard University Press.

Costa, J. L. (2016). *Atividades docentes de uma professora de Matemática: artefatos mediadores na EaD.* 2016. 213f. Tese (Doutorado em Educação). Universidade Federal de Minas Gerais. Belo Horizonte, MG.

David, M. M.; Tomaz, V. S. & Ferreira, M. C. C. (2014). How visual representations participate in algebra classes' mathematical activity. *ZDM*, *46*(1), 95-107.

Deodato, A.A. (2017). *Articulação entre disciplinas de uma escola de tempo integral: reverberações de um "Grupo de Trabalho Diferenciado (GTD)" nas aulas de Matemática.* 2017. 208f. Tese (Doutorado em Educação). Universidade Federal de Minas Gerais. Belo Horizonte, MG.

Engeström, Y.; & Sannino, A. (2010). Studies of expansive learning: Foundations, findings and future challenges. *Educational Research Review*, *5*(1), 1-24.

Fiorentini, D.; & Miorim, M. A. (1990). Uma reflexão sobre o uso de materiais concretos e jogos no Ensino da Matemática. *Boletim da SBEM-SP*, *4*(7), 1-5.

Lopes, A. R. L.V.; & Marco, F. F. (2015). Pesquisa em Educação Matemática e Psicologia Histórico-Cultural: alguns apontamentos. *Educação Matemática Pesquisa*, *17*(3), 456-471.

Lorenzato, S. O. (2006). *Laboratório de Ensino de Matemática na Formação de Professores.* Campinas, SP: Autores Associados.

Lüdke, M. & André, M. E. D. A. (1986). *Pesquisa em educação: abordagens qualitativas.* São Paulo, SP: EPU.

Matos, J. M. & Serrazina, M. L. (1996). *Didáctica da matemática*. Lisboa: Universidade Aberta.

Moreira, P.C. & David, M. M. S. (2005). *A formação matemática do professor: licenciatura e prática docente escolar.* Belo Horizonte, MG: Autêntica.

Pais, L. C. (1996). Intuição, experiência e teoria geométrica. *Zetetiké*, *4*(6), 65-74.

Pais, L. C. (2000). Uma análise do Significado da utilização de recursos didáticos no ensino da Geometria. *Anais da Reunião Nacional da ANPED*, Caxambu, MG, 23.

Ponte, J. P.; Brocardo, J. & Oliveira, H. (2003) *Investigações matemáticas na sala de aula*. Belo Horizonte, MG: Autêntica.

Rodrigues, F.C. (2011). *Laboratório de educação matemática: descobrindo as potencialidades do seu uso em um curso de formação de professores.* 2012. 195f. Dissertação (Mestrado em Ensino de Ciências e Matemática). Pontifícia Universidade Católica de Minas Gerais. Belo Horizonte, MG.

Smolka, A. L. B. (2000). O (im)próprio e o (im)pertinente na apropriação das práticas sociais. *Cadernos Cedes, 20*(50), 26-40.

Takinaga, S.S.& Manrique, A. L. (2022). A Teoria da Atividade como referencial teórico em pesquisas de ensino de conteúdos acadêmicos no contexto escolar inclusivo. *Revista de Produção Discente em Educação Matemática*, *11*(2), 64-77.

Van de Walle, J. A. (2009). *Matemática no ensino fundamental: formação de professores e aplicação em sala de aula*. São Paulo, SP: Penso Editora.