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Isometries: Epistemological Insights Among Secondary Students Interacting in a Dynamic Geometric Environment with Touches on Screen

Isometrias: Insights Epistemológicos de Alunos do Ensino Médio Interagindo em um Ambiente Geometria Dinâmica com Toques em Tela

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Marcelo Almeida Bairral
https://orcid.org/0000-0002-5432-9261
Federal Rural University of Rio de Janeiro
mbairral@ufrrj.br

Alexandre Rodrigues de Assis
https://orcid.org/0000-0002-3658-2829
Institute of Education Rangel Pestana – SEEDUC/RJ
profalexandreassis@hotmail.com

Abstract

This study investigates secondary students' understanding of isometry (symmetry, rotation, and translations) using dynamic geometry environment with touchscreen (DGEwT). The research question addresses issues concerning geometrical reasoning of Brazilian learners without previous knowledge regarding isometry. Data came from videotapes, register of learners' notes for each proposed task, icon sheet, and from screen recording. The analytical process was mainly based on the geometrical reasoning and related procedures. One designed task is shown and analyzed. Applying reflectional symmetry ideas, using procedures based on straight line and performing simultaneous touches on the screen, students achieved the composition of rotation and translation movement naturally and synchronically. The study highlighted the relevance of isometry in both school and teacher training curriculum.

Keywords: Isometry. Task-design. GeoGebra APP. Touch.

Resumo

Este estudo investiga a compreensão dos alunos do ensino médio sobre isometria (simetria, rotação e translação) usando ambiente de geometria dinâmica com tela sensível ao toque (AGDcT). A questão de pesquisa aborda questões relativas ao raciocínio geométrico de alunos brasileiros sem conhecimento prévio relativo a isometria. Os dados foram obtidos por meio de gravações em vídeo, registros dos alunos em tarefa proposta e na folha de ícones e de telagravações. O processo analítico baseou-se, principalmente, no raciocínio geométrico e procedimentos relacionados. Uma tarefa projetada é apresentada e analisada. Aplicando ideias de simetria reflexiva, utilizando procedimentos baseados em linha reta e realizando toques simultâneos na tela, os alunos alcançaram a composição do movimento de rotação e translação de forma natural e síncrona. O estudo destacou a relevância da isometria no currículo escolar e de formação de professores.

Palavras-chave: Isometrias. Design de tarefas. GeoGebra APP. Toques.

1. Introduction

Mobile devices with touchscreen, tablets, and smartphones have been taking an increasingly important role in the lives of individuals, in teaching and learning, and also in research processes. As tablets and particularly smartphones are extensions of our bodies, we have come to perform activities in our current lives which we would not have done without them. As a new language, touches on screen allow us other ways of feeling, thinking, and acting.

The history of humankind is continuously creating technologies, and these technologies, synergistically, keep re-dimensioning us. Our minds, bodies and physical environment (Moore-Russo & Viglietti, 2014) work in constant synergy. In the physical spaces we move through, we deal with technological, cognitive, cultural resources. Among those resources, devices such as smartphones or tablets provide, together with their mobility, a convergence of media in a single device, as well as ubiquity (the possibility to surf different spaces thanks to their connectivity)[1].

The possibility to make different constructions, to do simultaneous movements and adjusting by touch on screen seems to be a powerful resource for changing tasks as well as the nature of the geometric understanding concerning plane transformations using DGEwT. In the field of plane transformation, in general, users manipulate the screen using mainly one or two fingers and, sometimes, when working in pairs they can also share fingers or hands to manipulate some shape. Users can also interact with the device in three different ways: *with* the device itself (gyrating it in different positions, etc.) and interact *on* or *from* the screen. Whenever we refer generically to touchscreen manipulations, we include **(i) different ways of touching** the screen (single or double click, dragging, zooming, etc.), **(ii) handling the device itself**, **(iii) interacting from it**, or, **(iv)** possible gestures that can coexist in the interactive and **discursive scenario** (Bairral, 2020).

The handling we perform on a mobile device is a way to unveil and materialize our thoughts in a communicative act in order to favor an interaction. Interaction through mobile touchscreen basically occurs with the device recognizing and tracking the location of the user's input within the display area. This interactivity enables at least six types of touches (Arzarello, Bairral, & Dané, 2014): tap, double tap, long tap (hold), drag, flick, and multi-touch (rotate).

These types of motions (handlings and/or touching) have stimulated our research group[2] on mathematics education to study changings in geometrical thinking when users use dynamic geometry environment with touchscreen (DGEwT) to solve geometry tasks. This paper address results to the following research question: What implications does the use of tasks in DGEwT have on the reasoning and learning of isometries of secondary students (15-17 years old) without previous experience relating to this content and type of device?

In this paper our focus is on the analysis of manipulations performed by secondary students to carry out isometric transformations on a surface, as this has been little explored in a

[1] These aspects are detailed in Bairral (2019)
[2] www.gepeticem.ufrrj.br.

Brazilian context (Assis, 2020). From previous studies, as Arzarello, Bairral, & Dané, 2014, Bairral, Assis, & Silva (2015), Assis (2016), Bairral, Arzarello, & Assis (2017) and Silva (2017), we can ascertain the fruitful moment to continue to research on manipulations on touch-sensitive screens and the learning of isometries.

2. Motion, handling and understanding with DGEwT

Researching on learning in mobile environments of dynamic geometry (Arzarello, 2002; Arzarello et al., 2014) means taking into account perceptual aspects and considers not only objects (as constructions, for example) but the physical perception of learners, their movements, gestures, languages and the mediating artifacts that they use (and/or create). It is a way of perceiving, being, being in the world, a way of thinking. Through Paulo, Pereira & Pavanelo (2020), this way of seeing, i.e., this way of visualizing, gaming, touching what is shown by making itself known, is constituted in perception.

Two current Brazilian lines of research - guided by Merleau-Ponty's phenomenology - are enriching our perspective of embodied cognition with DGEwT, particularly on the issues of perception, spatiality, motion, visualization and understanding (Rosa, Farsani, & Silva, 2020; Paulo et al., 2020). Phenomenologically understood perception is a non-thetic and pre-reflexive experience perception which opens up, through lived experience, to the thoughts that are constitutive of the other, of myself as an individual subject and of the world as the pole of my perception (Rosa et al., 2020).

In the same way that we differentiate touch on screen from mouse click, Paulo et al. (2020) highlighted the difference between moving to see the movement of the projected object (using Augmented Reality app) and moving the object with a mouse. According to them the difference consists in considering the spatiality of the subject, of the person who moves and coexists with the objects in the environment, giving amplitude to the visual field. The authors added: "I, the person who moves my device, experience the vision, adhering to my gaze the virtual object that is projected in the physical environment through the screen of my smartphone, without breaking the link with the world" (Paulo et al., 2020, p. 653).

In the same way we differ touch on screen from click in mouse, Paulo et al. (2020) highlighted the difference between moving to see the movement of the projected object (using Augmented Reality app) and moving the object with a mouse. According to them the difference consists in considering the spatiality of the subject, of the person who moves and coexists with the objects in the environment, giving amplitude to the visual field. Through the authors we would emphasize: I, the person who moves my device, experience the vision, adhering the virtual object that is projected in the physical environment through (or with) the screen of my smartphone to my gaze without breaking the link with the world.

There are sensory devices that constitute and reproduce the player's bodily movements (which happen through player's avatar) and create a world in which mathematics can be perceived with other nuances (Rosa et al., 2020 2020).

When students play bowling the knowledge has been constituted in a long process of perception about the students' own bodies. The perception of movement finds meaning in the movement itself with the game. According to the researchers, the optimization of these very subtle movements that are shown in classrooms can have a direct positive effect, both on the teaching and learning process and on understanding students' perception of the mathematical task.

The Augmented Reality app also brings some possibilities of interaction with mathematical functions of the parabolic, circular, and sinusoidal type. The app used in Calculus allows the user to make it possible to drag, deform, transform, enlarge, or reduce images, and to highlight ways of seeing, handling, and experimenting, which can favour the development of the visualization process, starting from movement (Paulo et al., 2020).

In our sensorial process, motion and handling on screen take an important cognitive role and, in their movement into existence, in which they become objects of thought and consciousness, geometric concepts are endowed with particular determinations; they have to be actualized in sensuous multimodal and material activity (Radford, 2014). Taking into account the performance, i.e., the speed of the response from the device, when we execute manipulations on mobile devices only as touches on screen, it depends on three factors: **space**, that is, the area on which the touch is being performed or can be performed; **sustained action** of movement, with possible combinations of different types of touches; and **simultaneous** movement of several elements on the screen (Assis, 2020).

3. Isometries

In current Brazilian curriculum, isometries, as a secondary school subject, are proposed without much emphasis on digital technologies. Besides, they are thought with an emphasis on symmetry or on the sequence symmetry-rotation-translation, without approaching the composition (product) among them.

Delmondi & Pazuch (2018) remarks on the scarcity of Brazilian studies on the insertion of isometries in the courses for bachelor's degree in Mathematics. Assis (2020) signals the inexistence of tasks including concepts of isometries with DGEwT in Brazilian didactic books.

Silva & Almouloud (2021) presented a historic and epistemological work in which they stressed the importance of orthogonal symmetry being understood as a geometric transformation on a plane and a set with specific properties (the group of transformations). The mathematic object orthogonal symmetry, these researchers stress, should be understood in terms of its definition and its mathematic properties.

Based on this meagre field of research and innovative ways of instruction, the main reasons to focus our research on isometry with DGEwT are the following:

- **(i)** rotation and other gyrating movements on screen are often applied due to the various alternatives of handling touchscreen devices (Kruger, Carpendale, Scott & Tang, 2005; Tang, Pahud, Carpendale & Buxton, 2010);
- **(ii)** rotation and other plane transformations have remained unaddressed in Brazilian geometry classrooms so far;
- **(iii)** touchscreen devices provide possibilities of gyrating movements on screen, or with the device itself, which might result in new insights on embodied cognition;
- **(iv)** rotation and other plane transformations are concepts that involve intrinsically embodied motions;
- **(v)** symmetry, in particular, has a strong visual presence (in art, tiles, building structures etc.), and other ideas of balance and force present in our daily lives (gym equipment, for instance);
- **(vi)** dynamic conceptions of symmetry may enable students to develop more flexible spatial abilities when they engage in school geometry (Ng & Sinclair, 2015);
- **(vii)** besides the cultural and historic potential, through isometries we can explore other concepts (geometric and algebraic) and, through orthogonal (axial)

symmetry, it is possible to deduce all the isometries of the plane (Silva & Almouloud, 2021);

(viii) through perception of regularities in the moved (transformed) shapes, the study of transformations on a plane takes us to ideas of harmony and balance, also important to our brain for their potential to map regularities;

(ix) despite their absence in Prospective Mathematics Teacher Programs, plane transformations are important in understanding and deeper mathematical thinking in geometries (Euclidean and Analytical) and Algebras (Abstract and Linear).

The rotating movements appeared in the pilot study (2013), in which we wanted to analyze procedures and ways of touchscreen in tasks without a focus on isometries. Conceptually, in order to rotate one shape, we need to determine beforehand the center of rotation in each point, but with the use of two fingers the decision may not have been done beforehand. In task 1 (on Appendix A) Italian secondary students did rotation and other gyrating movements (with two fingers in movement, one fixed finger and the other in motion etc.) as detailed in (Arzarello et al., 2014). These movements have instigated us and, considering the relevance of isometry in mathematical thinking, we are going further in such analysis in our research project (granted by CNPq) to deepen our understanding of reasoning in the designed tasks focused on isometries in DGEwT.

It is important to consider DGEwT as a physical extension of our bodies (Bolite Frant, 2002), which interferes with our perception and way of being in the world (Paulo et al., 2020; Rosa et al., 2020). In this sense, we signal the importance of inserting touches and manipulations on screen into the semiotic package (Arzarello, 2006) as one more way of bodily cognition.

4. Materials and methods

In Brazil, even in secondary schools or Prospective Mathematics Teacher Programs, plane transformations do not appear in current official curricula. In mathematics education around the world, this absence is also observed (Berisha, Gimenez & Thaqi, 2013; Delmondi & Pazuch, 2018; Ng & Sinclair, 2015). Usually in Brazil, when plane transformations are taught, they are conceptually mapped in the following sequence: reflection symmetry (mainly identifying line of the symmetry in some figures), rotation and translation. The composition (product) of plane transformations is underexplored in geometry lessons.

The setting of teaching experiments carried out in our research team is summarized in chart 1. In all of them we work in public schools, with different sets of secondary students (15-17 years old) who have received no previous knowledge concerning isometry.

Chart 1: Information summarized from research carried out

Study	Settings basic information's	Main result(s)
Arzarello et al. (2014)	-Open task, students with previous knowledge on Cabri Geomètrie, but no experience in DGEwT -DGEwT used: Geometric Constructor	Students did rotation and other gyrating movements (with two fingers in movement, one fixed finger and the other in motion etc.).
Bairral et al. (2017)	-Tasks (pilot design) specifically designed for isometries	Two domains of handling: constructive and relational

	-DGEwT used: Geometric Constructor and GeoGebra with touch	
Assis (2016)	-Tasks (second version[3]) specifically designed for isometries -Students without previous experience in DGE and DGEwT -DGEwT used: Geometric Constructor and GeoGebra with touch	Students applied, intuitively, reflection, axial symmetry, rotation or translation concepts, isolated or mixed
Assis & Bairral (2019)		Students often applied axial symmetry based on mirroring ideas as a scaffolding concept during task solving
Assis (2020)	-Tasks (third version) specifically designed for isometries -Students without previous experience in DGE and DGEwT -DGEwT used: Geometric Constructor and GeoGebra with touch	-Students' types of touchscreen and detailed motion of fingers[4] -Students understood isometry as a transformation -Confirmation that students solve the tasks by making spontaneous composition between them in a non-linear reasoning. - Reflectional symmetry based on mirroring ideas, and compositions procedures observed in actions as to duplicate and to mirror, based on straight line

Source: Authors' own elaboration.

Each classroom-based intervention was 2 hours long and in each one the students worked alone or in pairs with a tablet or smartphone. The analysis process was mainly based on (i) the videotapes of students working on the software, (ii) written answers for each task, (iii) the use of one sheet in which he or she could write down and describe the function of each device icon (we named it icon sheet, Appendix H), and (iv) from screen recording, it allows the researcher to thoroughly observe manipulations which happen "within" the device (on the "backstage")[5].

Using the task below as example, in this paper we provide detailed results regarding students' reasoning, selecting icons and relating procedures, and understanding regarding isometry[6].

4.1 Task[7]: "Hey, with the straight line, you can turn!"

We are illustrating part of the interactions performed by the learners in the solution of task 7. With a planned 90-minute duration, the task aimed at the elaboration of a procedure in order to compose figure 2 from figure 1, using at least one of the four icons given in the task:

Open file "Task 07"

[3] The redesign of the tasks is not as linear as it seems. Appendix G shows the role of the process of designing the task throughout six years of data collection in regular public school classrooms. Some examples of tasks are presented in (Assis, 2017) and (Bairral, 2017).

[4] This typology can also be seen in (Assis, 2020).

[5] See more details on (Bairral et al., 2022).

[6] Our focus of analysis is on geometric reasoning, that is, transformation as a process that maintains (measure of the sides, angles and area, and shape) in the figure. For studies interested in transformation as a function, we recommend (Hollebrands, 2003).

[7] Designed from Veloso (2012).

Figure 1: Initial construction

From the construction presented in the file, represented in Figure 1, elaborate a procedure to build Figure 2, using at least one of the tools: 

Figure 2: Complete plate

Finally, describe the procedure that you guys used to your classmate.

"It looks too easy!" This was what Rose said when she opened the file with the initial construction in Task 7. Rose started negotiating with Maria, trying to establish a procedure for the construction of figure 2, as Maria, contrary to Rose, was trying not to use straight lines in her strategies.

Maria started looking for some icons (rotation, reflection, straight line, for instance) when trying to locate the resources from a previous task. Rose pointed at the center and asked if "four pieces form a perfect circle". The pieces she refers to are the circular sectors in the initial construction. The center of the sector coincides with the vertices of the square (Figure 1) and the radii overlap the adjacent sides of the square, which indicates that it refers to a sector of 90º, and "4 pieces" would be needed to form a circle.

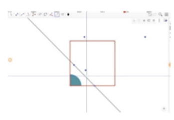
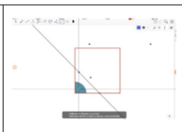
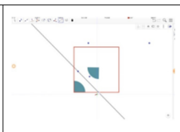
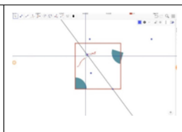
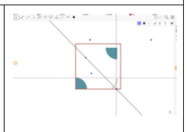
Rose: How are we going to do this?
Maria: Let me look at something here. [she selects the tool "erase" and she deletes two points that had been created during the construction]
Rose: Four of those pieces [she refers to the circular sector provided in the Figure 1], do they form a perfect circle?
Maria: What?
Rose: Four pieces like that?
Maria: I think so! Because, it's like… it's like this… it's like this… [she indicates by pointing to the circle at the center of the square on her task sheet]

During the time Rose was analyzing the construction on the task sheet, Maria selected the resource rotation in relation of a point, she created points in the working area, but then she decided to delete points on the screen. Rose stated that 16 "pieces" - referring to the circular sector (Fig. 1) - would be needed to compose the final figure. Maria gave up the attempt at using the rotation resource of GeoGebra. Rose started a new process and selected the icon reflection in relation to a straight line.

> **Maria:** There is the possibility to make a circle also.
> **Rose:** But do you want to start with the circle, or can it be here?
> **Maria:** Do that one there... [Rose carries on with the positioning of the sector]

Chart 2 illustrates the initial process of construction using the icon reflection in relation to a straight line.

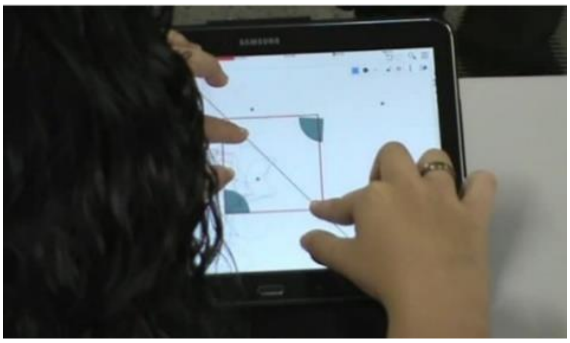
Chart 2: Cutting of procedure "Straight line that duplicates"

Time	4:41 ---------- 5:30					
Captured screen						
	(a)	(b)	(c)	(d)	(e)	(f)
Icon						
Type of touch						
Summarizing analysis	They selected "the straight line that mirrors" and touched on two different places, in the working area. Although one point was on the square, we did not identify any relations or use of it in the process of construction.	The idea was to "mirror" the sector in relation to the straight line built in a way to enable the positioning of the sector reflected indicated on the captured screen (f)	The learners selected the sector and then the straight line.	Reflected sector.	They chose a point and moved the other by rotating. In the captured screen (d) they performed a turn to the right and, when they realized that the radius of the sector would not be parallel to the side of the square, they slightly rotated to the left, while keeping contact with the screen of the device. Then - captured screen (f) - they selected the other point and with the circular movement to the left, they rotated the sector. This is how the adjustment took place. They changed the selected point and moved the sector until the center of the reflected figure coincided with the vertex of the square. This composition of rotations could have been done through translations.	

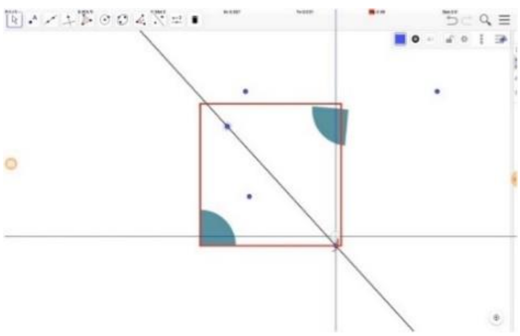
Source: Elaboration of the authors

When analyzing the videotape recording during the process of adjusting the reflected sector we could not identify whether the learner performed a double touch with fingers from both hands, as illustrated in figure 3 (a). When confronted with the screenrecording, it was possible to verify that Rose performed turns, selecting one point at a time (figure 3b), alternating the points in the straight line of reflection in centers of rotation. In this way, with a combination of circular manipulations, it was possible to perform the desired adjustment.

Figure 3: Selection of points on the straight line of reflection (5:27):(a) Videotape and (b) Screen recorder



(a)



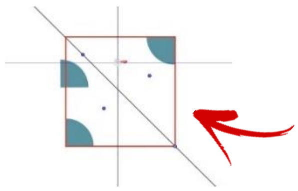
(b)

Source: Research data.

When she carried on with the procedure that had been started by Rose, Maria did some touches on the reflected sector and the line of symmetry, and she did not realize that the second reflection would be the sector of initial construction. At that moment, Rose stated: "We need to build another straight line".

Rose: I remember that we used one without need of the straight line. To the point. A reflection in relation to the point.

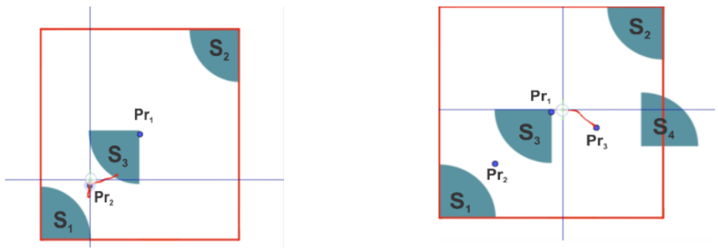
Maria: I used it, but you didn't like it. [She selects the resource, touches the sector and the internal area of the square]



Rose: But that one doesn't need a straight line. It just needs a point, see? Ah, Maria, we can put this one here. [Rose selects a point of symmetry and moves it in such a way that the reflected sector is positioned on the point indicated by her]

With the change of device, the students started a new construction. They did not start by using a reflection in relation to a straight line, but a reflection in relation to a point. In figure 4, we adopted S_n to indicate the order of the reflected setor Pr_n to indicate the point of reflection. The indication $\mathbf{S_x \rightarrow Pr_y \rightarrow S_z}$ means that $\mathbf{S_z}$ was obtained from the reflection of $\mathbf{S_x}$ in relation to $\mathbf{Pr_y}$.

Figure 4: Reflection of reflected sector: (a) $S_1 \rightarrow Pr_1 \rightarrow S_2$ and $S_1 \rightarrow Pr_1 \rightarrow S_2$; (b) $S_3 \rightarrow Pr_3 \rightarrow S_4$

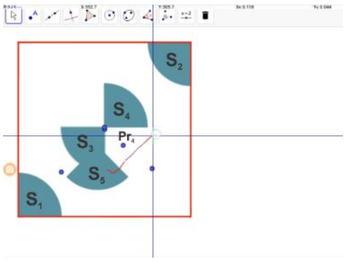
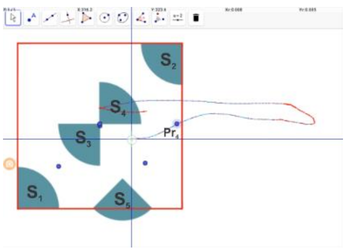
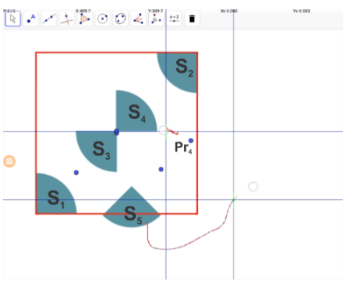


(a) (b)

Source: Research data.

The initial aim of the students was to construct the circle in the center of the square. So they used the icon rotation around a point. The choice of the icon rotation, the circular sector $\mathbf{S_3}$, the rotation point $\mathbf{Pr_4}$ and the indication of the 45º angle originated the reflected sector $\mathbf{S_5}$.

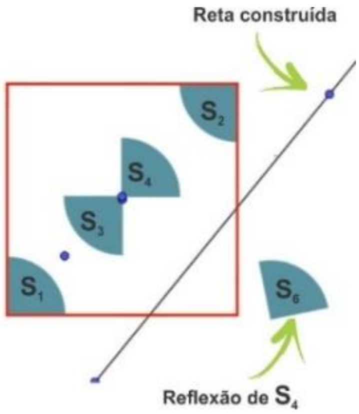
Chart 3: Manipulations trying to turn the reflected circular sector S_5.

Captured screen	Procedures	Touches
	The student touches the sector **S_5** and performs an upward movement on the right.	 She selects an element or construction and drags it upwards.  She selects an element or construction and drags it towards the right.
	She selected the reflection point (Pr_4) and shifted with the aim of turning sector S_5. Maria states: "We are using it but we are not getting it right".	 She selects an element or construction and drags it towards the left  She selects an element or construction and drags it towards the right
	Maria performed a movement that indicates that the rotation movement available on the device, and applied the adopted strategy by Rose when she used the mirroring straight lines. At some point, Rose exclaims that "with the straight line, you can turn!"	 While keeping one finger steadily fixed on one point, she rotates towards the right with another finger

Source: Elaboration of the authors.

After several attempts to rotate sector **S_5**, the students resumed the initial procedure, that is the "mirroring straight line". They started the process to obtain another sector reflected, which was named **S_6** (figure 5).

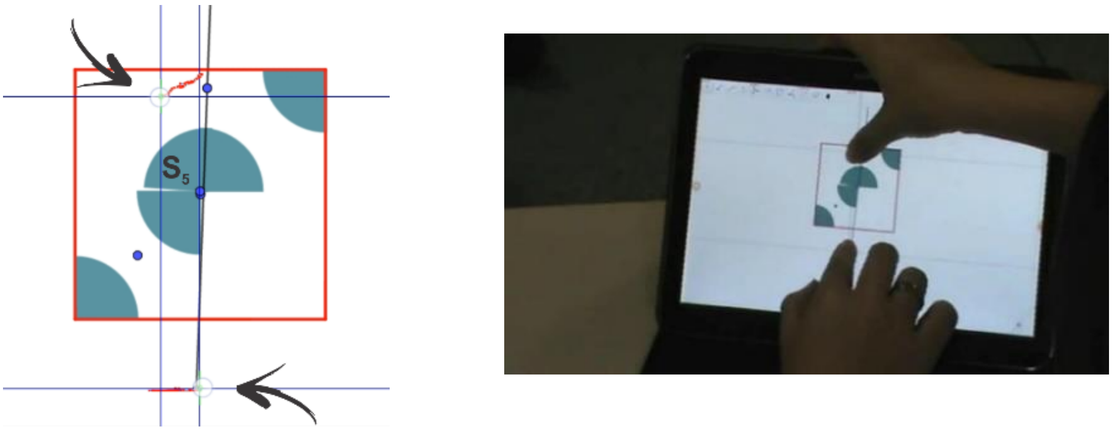
Figure 5: $S_4 \rightarrow$ Constructed straight line $\rightarrow S_6$ (Screenrecording.)



Source: Screen recording.

In figure 6, it is possible to identify two simultaneous touches (figure 6a), a handling performed by two students at the same time (figure 6b), two points on the constructed straight line.

Figure 6: Handling of the reflection line. (a) Screen recorder and (b) Videotape



(a) (b)

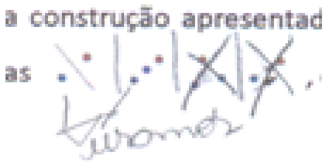
Source: Research data.

Such handling allowed the performance of necessary adjustments to compose the circle in the center of the square. From that moment, Rose and Maria realized that they could build a ceramic tile combining the icons reflection around the point and reflection about a straight line. Their ponderings revolved around finding out which icon was the most adequate to achieve their aim regarding which sector to reflect, whether it was in relation to either the straight line or the point, although all constructions were possible to be performed using just one of the forms of reflection allowed in the program. But that was not something pre-determined. At least one of the following icons was specified for the task.

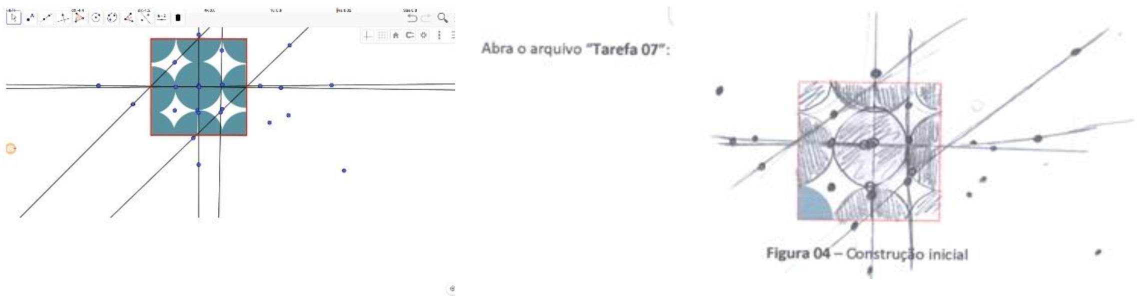
5. Results

The whole process of reasoning and solving the task is summarized in Figure 7. There, we can see the icons used to perform the construction and pictorial resources (on GeoGebra and in task sheet) to represent the final construction (the plate). The students' main reasoning was based on mirroring ideas, and compositions procedures observed in actions as to duplicate and to mirror, based on straight line (Assis, 2020) The straight line runs as scaffolding and supporting their understanding and procedures to do composition without linearity. By performing simultaneous touches on the straight line, students achieved the composition of rotation and translation movement synchronically. This synchronicity of touch and motions could be detailed and analyzed by screen recording (Bairral et al., 2022).

Figure 7: Inscriptions and constructions: (a) Identifying the icons used and (b) Illustration of the final figure



(a)



(b)

Source: Research data.

As requested in the statement of the task the students were invited to describe the procedure they had elaborated. They wrote down the icons they had employed. The difficulty was to determine the best sector to be reflected and the point of reflection. The use of straight lines to be mirrored was used, as the device allowed the manipulation with two simultaneous touches on the straight line, enabling the composition of rotation and translation movement synchronically.

Figure 8: Register of the procedure used on the Task Sheet "Ceramic plate"

We used and we duplicated the first, except we used the straight line, then we used in order to duplicate without the straight line. It was confusing, we changed tablets, and now we are using a different tool ( ). It was not right. We went back to using the first icons (and) . It started flowing again. We have most of the task ready, but the last ones are more difficult to duplicate or mirror in the right direction. Because of that difficulty, we had to recreate straight lines (because it is easier to

move with the straight line, we can use both hands). And last of all, after years trying without success, we managed to get them all done (that's right, ALL of them), all the circles.

Source: Research data.

The learners referred to the actions "duplicate" and "mirror" relating them to the resources  and , with a distinction of its geometric function for the obtained results. They indicated they had used the icon rotation, but they justified by saying that constructing straight lines and duplicating the shape was what they found to be the easiest way.

Those actions - duplicate and mirror - refer to procedures for combining isometries, powerful issues provided by the device and task design. This dynamic and non-sequential process (symmetry-rotation-translation) in relation to the use of straight line as a scaffolding concept, as they said, it was feasible because they could use both hands. This statement is also related to the type of possible touch to perform the combination rotation-translation in DGEwT.

It is important to point out that the students, after performing task 5 (Appendix E), in which they started pondering about their procedures, and task 6 (Appendix D), in which they identified ways to rotate the reflected triangle using the straight line and the manipulations available in the device, resumed the procedures used in previous activities in order to perform the construction combining manipulations on screen and resources in the device.

We cannot keep from stressing out the importance of the icon sheet (Appendix H) and the task sheet (Appendix F), as the students used them to pin down their remarks, establish relations from touches and quantify sectors. Task-design and other resources and ways of interactions used by teachers also take place in instruction and learning processes. In the illustrated example, the students' mathematical reasoning was enhanced through relevant processes such as conceptualization (isometries and plane transformation), composition, construction, and representation.

6. Discussion

Considering the relevance of isometries in the development of mathematical thinking, this article presents results on reasoning strategies among secondary students. From analyses carried out over six years of teaching experiments it was possible to observe that learners are able to solve the proposed tasks using DGEwT. There was unmistakable interest and patent involvement in the tasks on the part of all learners, even some who usually presented low scores in mathematics.

Understanding each isometry as a transformation, that is, a process that maintains the unchanged figure, regardless of position, is an important aspect of teaching (Veloso, 2012). In DGEwT a transformation requires a more dynamic approach in which a particular touch or handling is used to transform one initial figure into another. Thus, the teaching and learning of isometry requires tools, tasks and interactive opportunities that enable students to focus both on the action of plane transformations and the result of such transformations (Ng & Sinclair, 2015).

The analyses carried out throughout the studies summarized in Table 1 show that students do not apply the conventional sequence taught in Brazilian schools (symmetry → rotation → translation). It was an important finding, basically due to their weak background in geometry. Geometry taught in Brazilian schools, unfortunately, limits itself basically to calculus of perimeter, area from some known shapes and volume of some geometric solids.

The procedures adopted by students solving isometry designed tasks emerged naturally and without the traditional linearity in their reasoning. This non-linear reasoning was observed in all the proposed tasks. Freedom of movement, constructions and touches on the screen and other rotating performances favored by DGEwT should be considered in designing tasks. In our current analysis, we are checking whether leaving the task free, without icon indication, brings about any different results regarding geometric reasoning.

The option of choosing a tablet is another fact to be highlighted. We identified that this device possesses a peculiarity in relation to the *smartphone*, when it comes to manipulation possibilities, not to mention the question of available space for interactions, favorable to pair work. In the tablet, using GeoGebra as was our case, students identified a resource for manipulating a straight line, for example, with two fingers, which enabled them a composition of transformations, translation, and rotation in a synchronic way. For such an action, performed on a smartphone, using the same program, we have a "zoom" as an answer. It was only possible to identify such actions from the analysis of the videos generated by the App that recorded the touches on screen.

The tasks based on compositions of transformations favored the construction of a learning atmosphere, exploring various concepts of plane geometry without the need for the learners to have any systematic knowledge about the approached subject or the program that was being used. This allowed the learners to explore diversified ways in order to elaborate actions from the combinations of their own notes (based on outputs given by the tablet-software) with the possibility of touch screen for the resolution of tasks about isometries.

We can also highlight some implications for Mathematics teaching in didactic, cognitive, and epistemological dimensions. In the didactic realm we have the elaboration of mediating artifacts by potentiating the semiotic mediation and the importance of the use of DGEwT in a classroom environment, not as a verifying resource, but as an integral part of the process (Assis, 2020). In the cognitive dimension, we can find reasoning strategies related to manipulations on screen in elaborating mathematical meanings or procedures that have signaled reasoning. And in the epistemological realm, a rupture in the linearity of reasoning contributed to the construction of articulations which allowed for relations among rotation-translation-reflection.

Although these implications come from research with school mathematics and practice, they can contribute meaningfully to public policies of digital inclusion as well as ways of innovation in the production of research data with the use of the devices themselves or their applications.

As a new language, touches on screen allow us other ways of thinking and action, consequently, other possibilities of relating concepts, properties, motions, positions and procedures. This is a fertile and important field of mathematical reasoning involved in the instruction of isometries and other plane transformations. This article is also a call to the relevance of teaching this content in the school curriculum and in Prospective Mathematics Teachers programs.

This reflective process allows us to improve our mathematical learning in different ways, for instance, in geometrical (understanding and analyzing ways for doing composition of isometry), technological (knowing better devices' performances) and pedagogical aspects (designing different tasks considering device features).

Finally, we dedicate this article to the Native Peoples of the world and to the Brazilian Native Peoples in particular, whose lives and their very existence are threatened by the current government of Brazil. Native people use isometries brilliantly in their paintings and crafts. In the same way that we defend indigenous life, art and culture, we also value isometries in the development of mathematical thinking.

7. References

Arzarello, F. et al. A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik*, 34(3), 66-72.

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigación en Matemática Educativa RELIME*, v. 9, n. Extraordinario 1, 267-299. https://www.redalyc.org/pdf/335/33509913.pdf.

Arzarello, F., Bairral, M., & Dané, C. (2014). Moving from dragging to touchscreen: geometrical learning with geometric dynamic software. *Teaching Mathematics and its Applications, 33*(1), 39-51. https://doi.org/10.1093/teamat/hru002 .

Assis, R. de A. (2016). *Alunos do Ensino Médio trabalhando no GeoGebra e no Geometric Constructor: mãos e rotAções em touchscreen.* Master dissertation, Federal Rural University of Rio de Janeiro. Brazil. http://www.gepeticem.ufrrj.br/portal/wp-content/uploads/2017/04/Disserta%C3%A7%C3%A3o_Alexandre-Rodrigues-de-Assis.pdf.

Assis, A., & Bairral, M. (2019). Using touchscreen devices to improve plane transformation in high school classroom *International Journal for Research in Mathematics Education (RIPEM), 9*(1), 45-60.

Assis, A. R. de (2017). Transformações isométricas com manipulações em tela. *Boletim Gepem*(70), 186-197. https://doi.org/10.4322/gepem.2017.030

Assis, A. R. de (2020). *Alunos do Ensino Médio realizando toques em telas e aplicando isometrias com GeoGebra.* Doctoral dissertation, Federal Rural University of Rio de Janeiro. Brazil.

Bairral, M. (2017). Toques en pantalla y transformaciones planas. *Uno. Revista de Didáctica de las Matemáticas* (77), 46-50.

Bairral, M. (2019). Dimensions to Be Considered in Teaching, Learning and Research with Mobile Devices with Touchscreen. *Acta Scientiae, 21*(2), 93-109. https://doi.org/10.17648/acta.scientiae.v21iss2id5040

Bairral, M. (2020). Not Only What is Written Counts! Touchscreen Enhancing Our Cognition and Language. *Global Journal of Human-Social Science (G), 20*(5), 1-10. https://doi.org/10.17406/GJHSS

Bairral, M., Arzarello, F., & Assis, A. (2017). Domains of manipulation in touchscreen devices and some didactic, cognitive and epistemological implications for improving geometric thinking. In G. Aldon, F. Hitt, L. Bazzini, & U. Gellert (Eds.), *Mathematics and technology: a C.I.E.A.E.M source book* (pp. 113-142). Hamburg, Germany: Springer.

Bairral, M., Henrique, M. P., & Assis, A. (2022). Moving Parallel and Transversal Lines with Touches on Smartphones: A Look through Screenrecording. *The Mathematics Enthusiast (TME), 19*(1), 113-135. https://scholarworks.umt.edu/tme/vol19/iss1/7

Bairral, M., Assis, A., Silva, B. C. (2015). *Mãos em ação em dispositivos touchscreen na educação matemática*. EDUR.

Berisha, V., Gimenez, J., & Thaqi, X. (2013). *Using geometrical transformations in the primary mathematics textbooks in Kosova and Catalonia*. Paper presented at the CERME 8. https://www.researchgate.net/publication/324862054_using_geometrical_transformations_in_the_primary_mathematics_textbooks_in_kosova_and_catalonia

Bolite Frant, J. (2002). *Corpo e tecnologia*: Implicações para cognição matemática. (2002). http://www.ufrrj.br/emanped/paginas/conteudo_producoes/docs_25/corpo.pdf.

Delmondi, N. N., & Pazuch, V. (2018). Um panorama teórico das tendências de pesquisa sobre o ensino de transformações geométricas. *RBEP, 99*(253), 659-686. https://doi.org/10.24109/2176-6681.rbep.99i253.38616.

Hollebrands, K. (2003). High school students' understandings of geometric transformations in the context of a technological environment *Journal of Mathematical Behavior*, *22*(52-72).

Kruger, R., Carpendale, S., Scott, S. D., & Tang, A. (2005). *Fluid Integration of Rotation and Translation.* Paper presented at the Proceedings of the ACM Conference on Human Factors in Computing Systems (CHI)'05, April 2-7, 2005, Portland, Oregon, USA, pp. 601-610.

Moore-Russo, D., & Viglietti, J. M. (2014). Embodied cognition across dimensions of gestures. Considering Teachers' responses to Three-Dimensional Tasks. In L. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 137-227). New York: Information Age Publishing.

Ng, O., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM Mathematics,* 47(3), 421–434. https://doi.org/10.1007/s11858-014-0660-5

Paulo, R. M., Pereira, A. L., & Pavanelo, E. (2020). The constitution of mathematical knowledge with augmented reality. *The Mathematics Enthusiast (TME), 18*(3), 640-668. https://scholarworks.umt.edu/tme/vol18/iss3/11

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM – The international journal on Mathematics Education,* 46(3), 349-361. https://doi.org/10.1007/s11858-014-0591-1

Rosa, M., Farsani, D., & Silva, C. (2020). Mathematics education, body and digital games: The perception of body-proper opening up horizons of mathematical knowledge constitution. *Mathematics Teaching Research Journal*. 12 (2), 310-324.

Silva, C. V., & Almouloud, A. S. (2021). Um estudo histórico e epistemológico sobre o objeto matemático Simetria Ortogonal. *Intermaths,* 2(1), 71-87. https://doi.org/10.22481/intermaths.v2i1.7429

Tang, A., Pahud, M., Carpendale, S., & Buxton, B. (2010). *VisTACO: Visualizing Tabletop Collaboration. International Conference on Interactive Tabletops and Surfaces (ITS*

'10). Paper presented at the International Conference on Interactive Tabletops and Surfaces (ITS '10), Saarbrücken,Germany.

Veloso, E. (2012). *Simetria e transformações geométricas*. Lisboa: APM.

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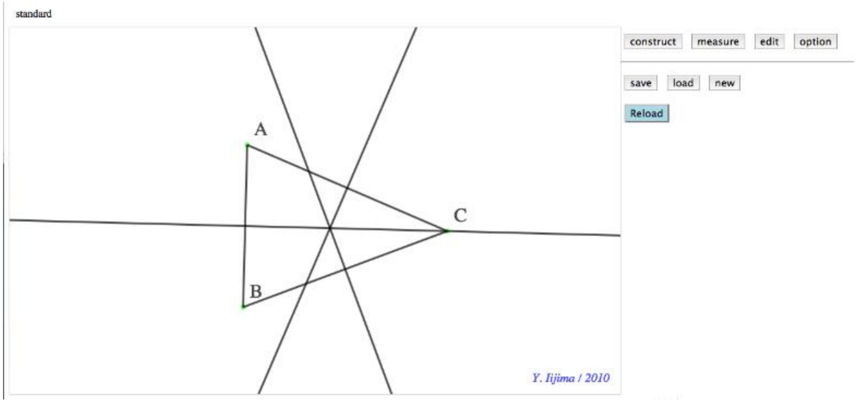
Appendix

Appendix A: Task 1.1 – For introduction and familiarization with Geometric Constructor device (30 minute)[8]

Use the software commands (construct, measure, etc.) to understand their functions, then draw the triangle using the commands on the iPad; write your remarks. Before exploring the software write down two observations:

Figure 1: Screen from GC.



Conceptually, in order to rotate one shape we need to determine each point (the center of rotation) beforehand, and with the use of two fingers the decision could have not been done beforehand. This type of action was not explicit for students exploring task 1.1. We became intrigued and we are investigating new conceptual aspects for the way we deal with rotation and other gyrating movements (with two fingers in movement, one fixed finger and the other in motion etc.).

[8] Links where to find the software and this activity:
a) with PC http://www.auemath.aichi-edu.ac.jp/teacher/iijima/GChtml5/GChtml/ server_e/gc_00026-test.htm.
b) with I-pad:2012/10/10 16:39 482434 gc_00026-test.htm

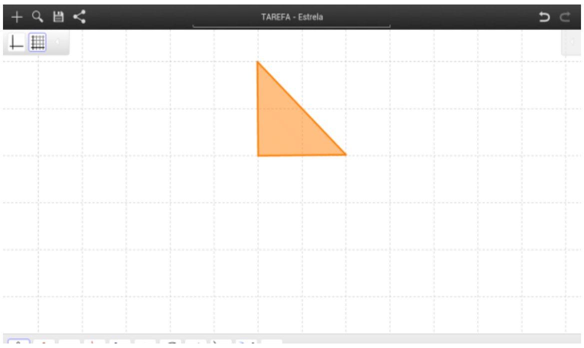
Appendix B: Task 2.2 (design 1) – Star task

Using only triangle rectangle and isosceles construct the following picture.

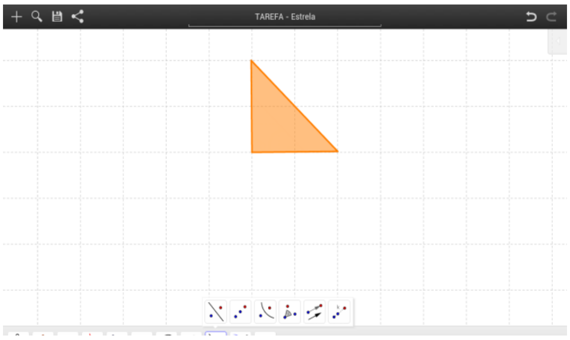
Now, write to a friend and tell him or her how you constructed the picture. When solving task 2.2, which involved the concept of rotation and using a device with a single touch, we observed that students used their fingers – no more than two – in a similar way to what students did when dealing with software Geometric Constructor in task 1.1, which did not apply to the referred concept. Although the task 1.1 had been designed (without a specific geometric concept) for free exploration and to know the software, the students made a lot of interesting gyrating movements. After observing such way of manipulation, we designed a set of tasks (see task 4.5 below), for which students had to apply the concept of rotation and other plane transformations.

Appendix C: Task 4.5[9] (design 2 from task 2.2) – Star task[10]

Open the file "Star task". Only the following triangle will appear:



Selecting the tool  a bar will open with 6 options:



Elaborate a strategy to construct the following picture using only the tools   

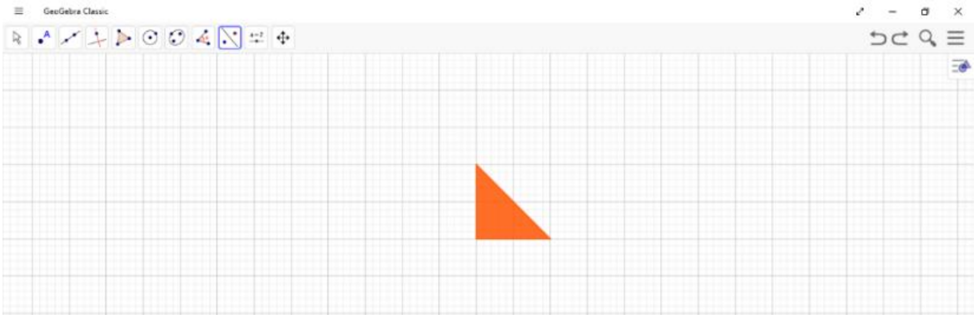
[9] Access https://drive.google.com/file/d/0B6zQPvF8JeJcMHoxUkJ1eTh2UGM/view?usp=sharing to see the video recorded by Adriano solving task 4.5 as discussed in Bairral et al. (2017).

[10] This version restricts the use of icon.

Appendix D: Task 6 (design 3 from task 4.5) – Star task (dez./17)

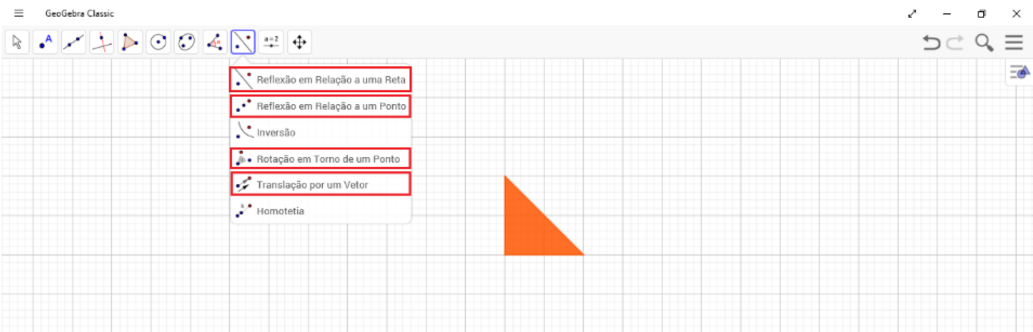
Open the file ***"Task 06"***. Only a triangle will appear as shown in the next figure:

Figure 1: Home Screen.



Selecting the tool will open  another bar with 6 options, observe the highlighted tools:

Figure 2: Screen indicating the allowed tools.



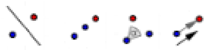
Using at least one of the tools  , elaborate a strategy to build the "Star" represented in Figure A3.

Figure 3: Star

(Write down for a friend all the procedures used to make the construction of Figure 3).

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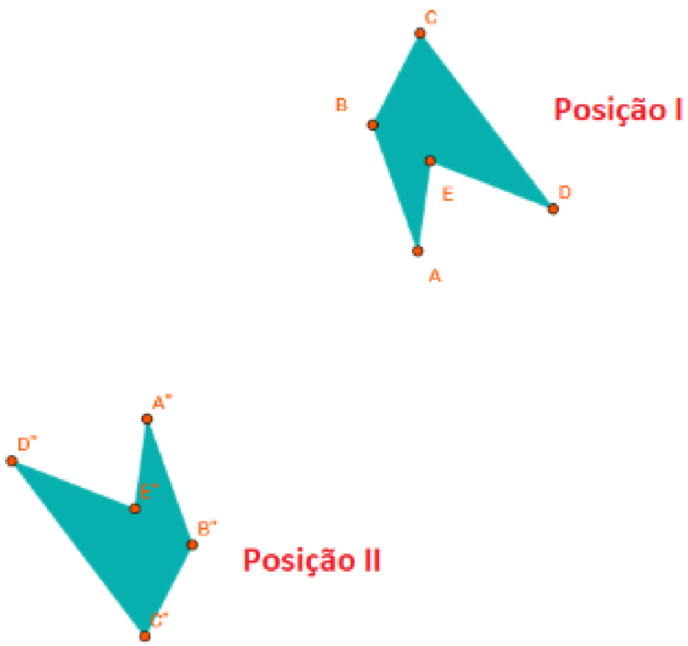
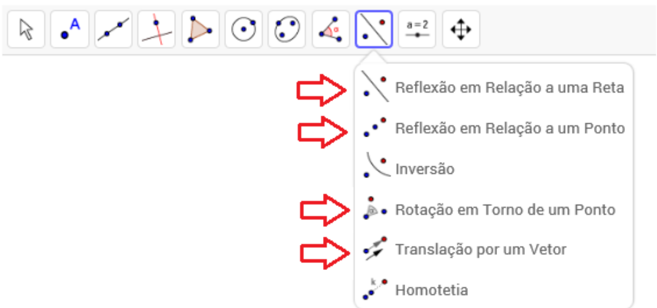
Appendix E: Task 5 – Shifting the polygon (dez./17)
Open the file "***Task 5***"
We will have the following construction

Figure 1: Shifting the polygon.



Note the tools highlighted with an arrow, represented in the next figure:

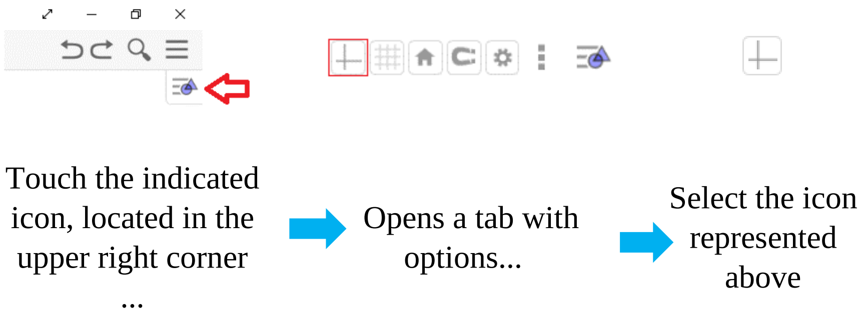
Figure 2: Toolbar.



Use at least one of the tools highlighted and develop a strategy to move the polygon position I to position II, knowing that it is permissible to use other GeoGebra tools. Describe your strategy.

If deemed necessary, you can insert the coordinate axes. To do this, follow the guidelines below:

Figure 3: Selection in the toolbar.



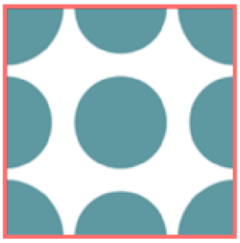
Is it possible to perform such a shift using a single tool only once? Justify.

Appendix F: Task 07 – Ceramic tile (dez./17)
Open file "Task 07"

Figure 4: Initial construction.

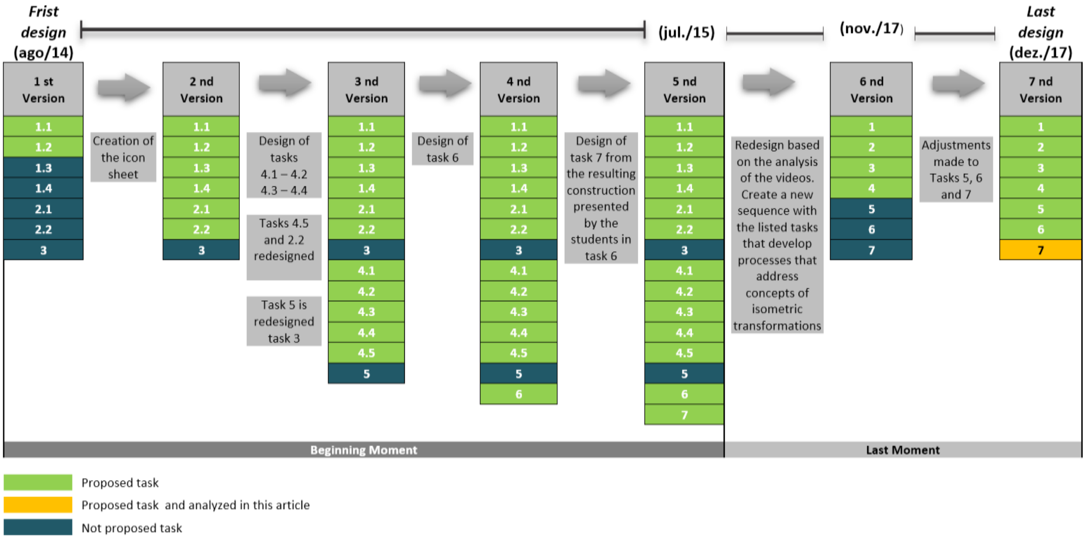
From the construction presented in the file, represented in Figure 4, elaborate a procedure to build Figure 5, using at least one of the tools: 

Figure 5: Complete tile.



Finally, describe the procedure that you guys used to your classmate.

Appendix G: Proposed tasks and the (re)designing process



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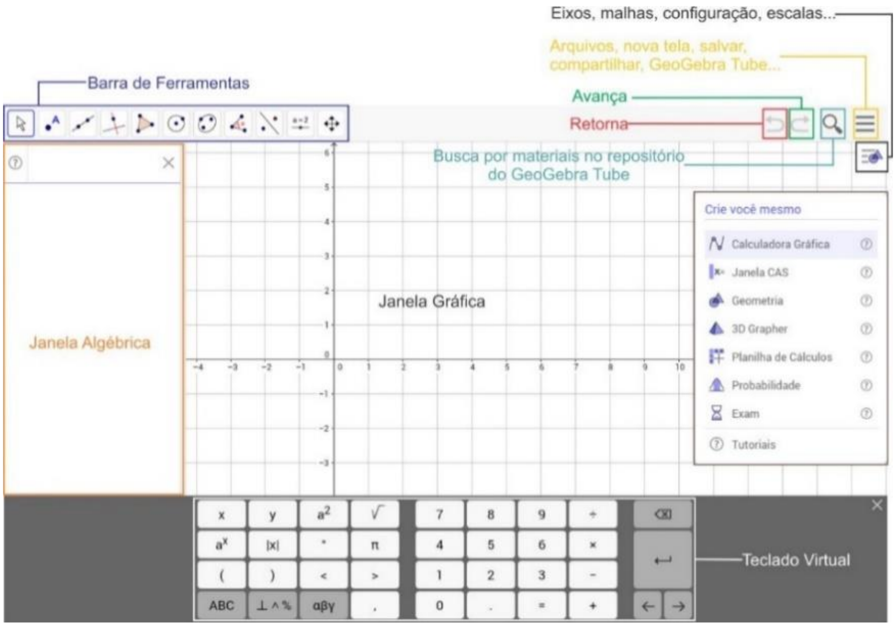
Appendix H: Icon sheet

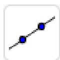
Folha de ícones do GeoGebra

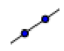
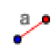
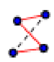
Tela inicial (Opening screen)

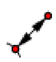


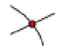
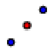
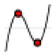
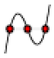
Barra de ferramentas (Tool bar)

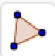
	
	
	

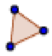
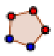
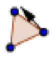


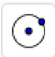
	
	
	
	
	
	
	

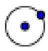
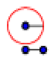
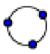
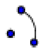
	
	
	

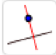
	
	
	
	
	

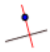
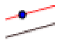
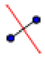
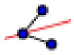
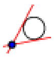
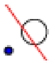


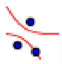
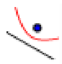
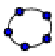
	
	
	
	

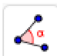


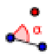
	
	
	
	
	
	
	
	
	



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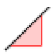
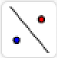
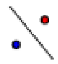
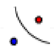
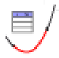
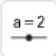
	



	
	
	
	
AA	
	
	

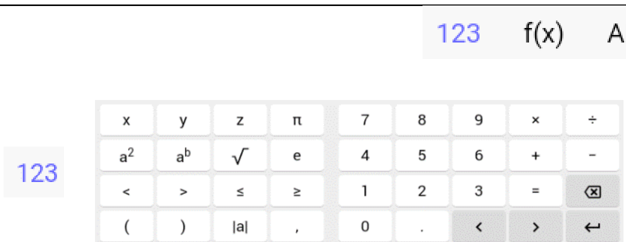
{1,2}	
$a \stackrel{?}{=} b$	
	



	
ABC	
	
OK	
	
a = 1	

Telas Auxiliares (Auxiliary screens)

 Teclado Virtual (Virtual keyboard)



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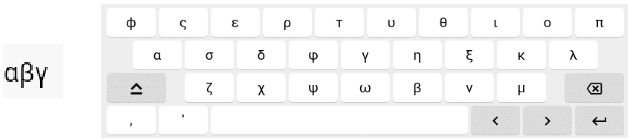

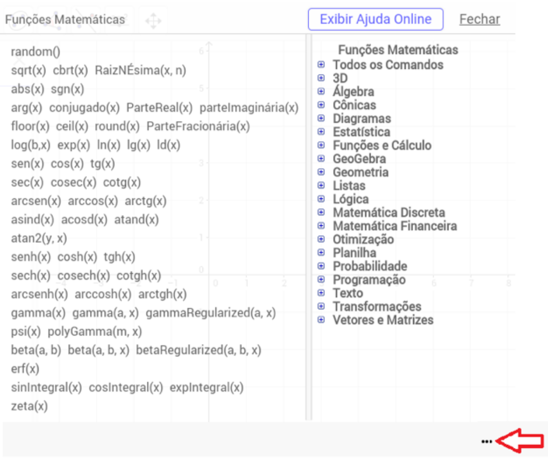

[Tela inicial: opening screen

In Appendix G, it should read: 1st, 2nd, 3rd, 4th, 5th, 6th, 7th version]