PRODUCTIVE DISCUSSIONS FOR ALGEBRAIC THINKING: GENERALIZATION AND JUSTIFICATION CONTEXT

DISCUSSÕES PRODUTIVAS PARA O PENSAMENTO ALGÉBRICO: CONTEXTO DE GENERALIZAÇÃO E JUSTIFICAÇÃO

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ABSTRACT

As emphasis on mathematical reasoning—defined as making and justifying conjectures —grows internationally, the need for studying classroom practices that balance the support of reasoning and discourse is also growing. This qualitative study reports teaching practices that supported students' mathematical reasoning during a teaching experiment in a rural fifth grade classroom. Particularly, it focuses on the teacher's reasoning and observed practices when planning, facilitating small group and whole class discussions, and the mathematical reasoning co-constructed within such practices. Connecting verbal and symbolic generalizations, recursive and explicit generalizations, and purposefully sequencing responses and tasks in terms of sophistication and understandability are some of the discursive practices that emerged from the data as supports of mathematical reasoning. Audio and video recordings of classroom activities, teacher's reflections, and observation notes were data sources. Implications for research and practice are discussed.

Keywords: Discourse, algebraic thinking, discussions, elementary school.

RESUMO

À medida que a ênfase no raciocínio matemático - definido como fazer e justificar conjecturas - cresce internacionalmente, a necessidade de estudar práticas de sala de aula que equilibram o apoio ao raciocínio e ao discurso também está crescendo. Este estudo qualitativo relata práticas de ensino que apoiaram o raciocínio matemático dos alunos durante um experimento de ensino em uma sala de aula rural do quinto ano. Particularmente, ele foca no raciocínio do professor e nas práticas observadas ao planejar, facilitando discussões em pequenos grupos e em toda a turma, e o raciocínio matemático co-construído dentro de tais práticas. Conectar generalizações verbais e simbólicas, generalizações recursivas e explícitas, e propositadamente sequência respostas e tarefas em termos de sofisticação e possibilidade de compreensão, estão algumas das práticas discursivas que emergiram dos dados como suportes ao raciocínio matemático.

Gravações de áudio e vídeo de atividades em sala de aula, reflexões do professor e notas de observação foram fontes de dados. Implicações para pesquisa e prática são discutidas.

Palavras-chave: Discurso, pensamento algébrico, discussões, ensino fundamental.

1. Productive Discussions for Algebraic Thinking: Generalization and Justification Context

Bergqvist and Lithner (2012) defined mathematical reasoning as “the line of thought that is adopted to produce assertions and reach conclusions when solving tasks” (p. 253). Mathematical reasoning may refer to the thinking processes, the product of those processes or both. The product may be the verbal or the written transcripts of students. From this definition, mathematical reasoning in this study refers to making and justifying generalizations about pattern finding tasks. Specifically, the mathematical reasoning focus is on the product of the thinking processes—students’ written and verbal generalizations and justifications.

Mathematical generalizations in pattern finding activities, according to Lannin (2005), may be classified as recursive or explicit. Recursive generalizations use the term-to-term change in the dependent variable to find unknowns. Explicit generalizations relate the dependent and independent variables and enable calculation of outputs given *n* inputs without necessarily knowing the previous outputs. Sowder and Harel (1998) classified justifications as externally based schemes, empirical schemes and analytic schemes. With externally based schemes, students do not show ownership of the justifications but instead refer to an external source, which may be a book or another person perceived as more knowledgeable. Empirical schemes show students’ ownership of the justifications but do not regard the generality of the context. Analytic justifications consider the generality of the mathematical task’s context.

Increasingly, researchers and educators recommend that mathematical reasoning should be the core of any branch of mathematics education (e.g., Martin & Kasmer, 2010; Sowder & Harel, 1998). Mathematical reasoning is also a focus of national standards for teaching mathematics (National Council of Teachers of Mathematics [NCTM], 2000; Common Core State Standards Initiative, 2011). Creating opportunities for mathematical reasoning serves many purposes in mathematics education. It supports conceptual understanding and retention of students in advanced mathematics classes (Martin & Kasmer, 2010; Horn, 2008). Moreover, students’ understanding is dependent on opportunities to reason mathematically (Bergqvist & Lithner, 2012). Despite such importance, research has shown that most students have difficulties reasoning mathematically (Lannin, 2005; Ellis, 2007; Healy & Hoyles, 2009).

“One of the most reliable findings from research on teaching and learning is that students learn what they are given opportunities to learn” (Hiebert, 2003, p. 10) since learning is “gaining access to a certain discourse” (Sfard, 2001, p. 160). Based on NCTM’s (1991) definition of discourse, teachers’ discursive practices include ways of representing ideas and the reasoning involved, ways of interacting with students, and values in particular interactional systems. There is ample research showing that, despite reform efforts that promote rich discussions whereby students use each other’s ideas as thinking tools to co-construct new understandings, univocal discourses in which teachers typically evaluate teacher-initiated responses are the norm (Cazden, 2001; Good, 2010; Temple & Doerr, 2012). Furthermore, research in US classrooms shows that about 80% of classroom time is used up by teacher speech in univocal discourses (Wertsch &

Toma, 1995). In such discourses, students do not actively use peers' ideas as thinking tools by evaluating the extent to which they are valid and/or connect to their own reasoning. "For teachers to use discourse effectively in mathematics instruction, they must understand what they are doing, how they are doing it, and how it influences learning" (Truxsaw, Gorgievski, & DeFranco; 2008, p.67). One can reasonably argue then that prevalence of univocal discourses is a reflection of impoverished knowledge about productive discursive practices, which results in difficulties in mathematical reasoning. Hence there is a need for research focus on discursive practices that support mathematical reasoning and teacher's reasoning behind their practices (Bastable & Schifter, 2008).

In response to this problem, (project name blocked) research team analysed elementary school students' generalizations and justifications and noted that in one classroom, students' mathematical reasoning progressed more easily relatively. As presented in table 1, the percentages of students using each class of generalization indicate that students tended to use explicit generalizations by the third day of our teaching experiment. Students also progressed to using empirical and analytical justification schemes, with more students using the latter, as shown in Table 2. Since it is a challenge to support development of mathematical reasoning (Baxter & Williams, 2010; Sherin, 2002), and opportunities for students to reason mathematically are tied to discursive practices (Imm & Stylianou, 2012), the discursive practices in this particular classroom—which based on the research team's experience seemed different from other classrooms—became the focus of the study. The following research question guided this focus: What are the discursive practices that supported students' progress of mathematical reasoning during this teaching experiment?

Table 1

Percentage of Different Generalizations Expressed by Fifth Grade Students

Generalization strategy	**Day 1**	**Day 2**	**Day 3**
Recursive	69.6	30.7	8.9
Explicit	30.4	69.3	91.2

Table 2

Percentage of Different Justification Schemes Used by Fifth Grade Students

Justification strategy	**Day 1**	**Day 2**	**Day 3**
Externally based schemes	13.0	6.2	0
Empirical schemes	39.1	43.8	8.7
Analytical schemes	47.8	50.0	91.3

2. Conceptual Framework

In this study, learning is viewed as an individual's construction of knowledge through participation in communities that create opportunities for appropriating tools that may modify abilities and dispositions (Claxton, 2002; Sawyer & Greeno, 2009). This perspective calls for "analyses focused on coordination of actions of individuals with each other and with material and informational systems" (Anderson, Greeno, Reder & Simon, 2000, p.12). As may be noted, one methodological implication of this perspective is a focus not only on students' mathematical understanding, but also on how teachers coordinate the classroom activities that mediate students' understanding. Based on a review of literature, Stein, Engle, Smith and Hughes (2008) proposed five practices (anticipating, monitoring, purposeful selection, sequencing, and connecting student ideas) that may afford coordination of students' ideas with each other, and with mathematical ideas in ways that potentially support growth of mathematical understanding. The following practices, support productive discourse.

Anticipating is the teacher expectation about students' possible interpretation of the problem, representations, and solution strategies. To monitor student responses, the teacher moves around the classroom to take note of student responses as they work on their tasks. A teacher should then purposefully select strategies, ideas, and representations for display during whole class discussion. Selected responses should be sequenced purposefully during whole class discussions for optimal attainment of lesson objectives. Class discussions present a chance for teachers to guide students to connect ideas and strategies from their peers. Skillful questioning and sequencing of tasks provide opportunities for connections.

As stated earlier, these practices were discussed by different authors. Stein et al. (2008) integrated these practices, showing their dependence on each other, and discussed how they may support mathematical understanding using hypothetical classroom situations. Readers are encouraged to read Stein et al. (2008) for a more thorough discussion of these practices. The present study contributes to this framework and mathematics education by reporting how these practices, integrated as in this model, can be enacted in an elementary mathematics classroom and possible teachers' reasoning in making the decisions regarding the practices. This paper also gives evidence of how these five practices support mathematical reasoning (see Tables 1 & 2).

3. Methodology

3.1 Research Design

This qualitative study reports findings from a teaching experiment. Teaching experiments draw from prior research to build on empirical results and are characterized by their goals (e.g., explore student reasoning), their interventionist nature and their use of combinations of data analysis techniques (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Steffe & Thompson, 2000). This teaching experiment was conducted in a fifth grade classroom in a rural county in the South Eastern US. The goal was to explore students' generalizations and justifications about patterning tasks. It was conducted over a period of 3 consecutive days. Each lesson took about 90 minutes on each day. One of the researchers, referred to in this study as the teacher, replaced the classroom teacher during the teaching experiment. A team of 6 researchers collected data during the lessons. After the first and second day of the teaching experiment, the research team met to discuss the critical events that supported student reasoning and how reasoning could have been supported further. Studying the teacher's discursive practices was a post hoc after noting students' progress in reasoning.

3.2 Instructional Tasks and Data Analysis

Students were seated in pairs and worked on trains table tasks adapted from Phillips, Gardella, Kelly and Stewart (1991) as in figure 1 and Table 3. All students' written work was collected at the end of the lesson. A video camera or an audio recorder focused on each pair. Whole class activities were video recorded. All classroom activities were transcribed. The teacher read the transcripts and wrote reflection notes on what her reasoning behind classroom practices was. Line by line coding of the transcripts was conducted. Active verbs were used to describe activities in the line-by-line coding. When building themes from line-by-line coding and the teacher's reflections, discursive tools from several frameworks were tried. Stein et al.'s (2008) framework was preferred among other frameworks because it explained most of the data. It should be noted that the teacher was not intentional about using Stein's framework in her teaching, her practices just happened to align with the framework during data analysis. The five practices were the broad categories, which were a basis for the 50-minute open-ended interview with the teacher. Within each theme, classroom events were randomly selected, and the teacher was asked for her interpretation of those events and reasoning for her classroom practices. Interview transcripts were analysed using the broad categories as the descriptors of practice. Data within each broad category was analysed and narrow themes developed. Descriptors for the narrow themes were from the data. For example, *connecting* was a broad category from Stein et al.'s framework, whereas connecting recursive and explicit rules, and connecting ideas across tasks, were corresponding narrow categories that emerged from the data. This analysis showed both the conditions to (e.g., reasoning behind practice) and the outcomes of the practices (e.g., how the practice supported students' understanding). The conditions and outcomes were analysed to provide a deeper understanding of the practices and to seek evidence that the practices co-constructed students' reasoning.

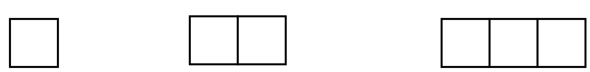
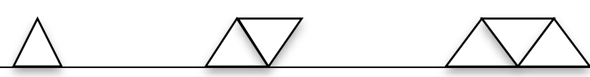
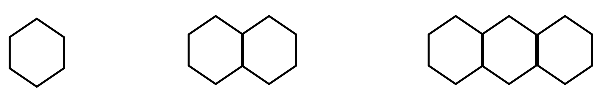
Square Tables Train	
Triangle Tables Train	
Hexagon Tables Train	

Figure 1. Sample of Train Tasks Given to Students

Table 3
Students' Summary Worksheet

Suppose I wanted to find the NUMBER OF CHAIRS for any polygon table trains. Can you find a pattern that will help you write a rule that works for any polygon?

	Trains with Triangles	Trains with squares	Trains with Pentagons	Trains with Hexagons	Train with n-gons
Rule					
Explanation					
General rule to find number of chairs for any shape train					
Explanation of the general rule for any shape train					

The study used different data sources and multiple coders. Given a set of data and a list of themes, another member of the research team coded the data and confirmed coding consistency. Since this was a directed doctoral study, at least 2 experts in the field of mathematical reasoning checked the coding scheme at each phase of the analysis. These practices built consistency and trustworthiness into the study (Creswell, 2007).

4. Results

Data analysis results showed that anticipating, monitoring, purposeful selection, purposeful sequencing and connecting were the teacher's discursive practices that supported mathematical reasoning. These practices will be discussed in detail in the following sections. Episodes, which are excerpts from the data that were selected based on how clearly they represented the discursive practices, will be used in reporting the results.

4.1 Anticipating

The teacher explained in her written reflection in that she had used the task before with students older than the fifth graders. From that experience, she noted that finding and justifying generalizations about the number of people who can sit around a train of triangle tables is more challenging because the patterns are not as visually explicit as with the rectangle and other tables. She explained in her written reflection:

> *I have learned from the first teaching experiments that triangle tables are difficult for students to find an explicit rule so I like to start the pattern block tables with squares. Generally, students at all levels start with a recursive strategy.*

As such, she decided to give the students the square table task before giving them the triangle table task. Furthermore, she anticipated that some students' reasoning would be based on the observed action as a table is added to the existing train to produce a longer train. She also anticipated other students would reason by considering the varying or constant aspects from train to train. Additionally, the teacher anticipated that students could use strategies that she had not come across before. This thinking can be noted in the following interview excerpt:

> *What I learned was that there are ways that people perceive the trains (task). Some people perceive the last action of building the trains and they forget about parts of the train that came before. Then there are those that focus on the whole train and the ends. Then there are those that focus on the middle and the two ends.*

Her previous experience with teaching the train tasks helped in anticipating what the students' responses might be. What she anticipated influenced her planning stage as she made decisions on how to sequence the tasks for optimal reasoning by students. She also anticipated that the students might not have had a lot of experience with finding patterns. As such, she planned to spend considerable time to introduce pattern-finding activities and the language (e.g., building models, input/output tables, justification, and rules or generalizations) associated with such activities.

4.2 Monitoring

As students were working on the tasks in pairs, the teacher walked around the classroom to note students' responses. This gave students a chance to ask clarifying questions about the tasks and for the teacher to ask probing questions to support and assess students' reasoning as explained in this interview response

> *I have a strategy where I walk around while the students are working in their small groups. I talk to them about what they are doing. There are several purposes why I do that. It is formative assessment. To see how far along in the reasoning they are or how quickly they are completing the task, or the different strategies they are using.*

Monitoring was also observed in the classroom. During the second day of the teaching experiment, the teacher was monitoring as students worked on the triangle tables train. She encouraged them to think of different ways of justifying their rules.

Teacher: (to student 1 and 2) So, have you figured this out yet (how to justify $t+2 = p$ as a rule for finding the number of people who can sit around a train of *n* triangle tables)?
Pat: The two comes from the two more people.
Taylor: From the two sides.
Teacher: Which two sides?
Taylor: Actually from the two sides of the triangle that you add to.
Teacher: Okay, build a model and show me where those two are.
Taylor: Well, when you have a three-triangle table and then you add another triangle to it. There are two extra sides.
Teacher: Okay. All right. There's another way of thinking about it, too.

Monitoring created an opportunity for the teacher to note students' reasoning and responses. Other questions aimed at challenging students to move beyond the recursive rules they expressed to developing explicit rules. Monitoring students' responses also informed the teacher on how to effectively progress with the lesson. For example, while thinking aloud during the lesson on the square table task, the teacher stated, "I will wait until they build their models and then we will talk about the t-table or the input-output table." Although she had planned to spend ample time discussing how to collect data into an input-output table, monitoring students' responses redirected that plan. In her reflection, she explained, "I decided to abandon giving directions on building the t-table. Brenda and Dan already understood how to collect their data." After noticing that some students already knew how to build input-output tables, it was observed that she briefly discussed the tables. This was done to make sure all other students in the classroom understood how to collect data using t-table and for the students to "learn how to collect data in a systematic manner."

4.3 Purposefully selecting responses

To support understanding, responses for public display were purposefully selected. The teacher considered several factors.

> *Then I want to pick out the surprising reasoning. I also want to give every child if I can, or every child who wants to, a voice in the large group discussion... They are very proud of what they developed. It does not matter how sophisticated or how unsophisticated… When this student used symbols to express her rule, I saw many recursive rules. And so in that case I would pick them (both recursive and symbolic) to go up… and then any different ways of expressing the generalizations. I would put those up (and say) oh you had another way. So I would want to say, look at all these different ways that we can do this (interview).*

This interview response shows that she purposefully selected a surprising response that she did not anticipate or a response that was not common in the classroom. She also explained that she selected responses to allow many students to present their ideas. Additionally, the teacher considered responses that were expressed differently to communicate to the students that the reasoning tasks could be approached using different strategies and expressed differently.

4.4 Purposeful sequencing of responses

Student responses that were selected were purposefully sequenced to optimize algebraic reasoning. During whole class discussions, she generally ordered presentation of students' strategies from less to more efficient. For example, recursive rules were generally followed by explicit rules as she explained. This sequencing may be observed in this whole class discussion on day 1:

> *King*: I did my table (shows his t-table to the class on the document camera). And every time you add a table (square), you add two people because when you put a table to tables (add a table to the existing train), you can't like this–you can't put anybody right here (on the touching sides of the tables), so you can only put two here (pointing at the top of the model) and here (pointing at the bottom of the square table model).
>
> *Teacher*: So you can only add two people?
>
> *King*: Usually you can add four, one, two, three, four, (points at seats on a model of one square table) but you can only have six there when they are like that (indicates his model of two tables).
>
> *Teacher*: Okay. So we have got Brenda. Brenda, do you want to come up and share?
>
> *Brenda*: (Brenda puts her work on the document camera for the class to read) I did if you add one table, you add two chairs and then I did T, which is the tables, times two, plus two equals chairs.

King explained his recursive rule, followed by Brenda, who had an explicit rule. Purposeful sequencing of responses was achieved by having whole class discussions in multiple phases. There were three main phases. Phase one discussions were after or when students built the models and collected data into the input/output tables. Phase two was after students explored patterns and were prompted to respond to questions that could be answered using recursive rules. Finally, whole class discussions were held after students were challenged to develop explicit rules by looking for a rule that could be used to find outputs for *any* input or for an input of 100. This practice created a context in which recursive rules were presented before explicit rules were.

The teacher also considered the strategies and solutions that could be understood by most students to be presented first and those that could be relatively difficult to understand. For example, Samantha explained

> *Samantha*: If you figure out how many people can sit around 20 tables, and you add 20– because 10 is 22 and if you add 20, that is 42, and if you add 20 to that, that's 62

> and that figures out how many people fit around 30… because if you just multiply (102 for 50 tables) by two you are adding another two end chairs.
>
> *Alex*: You have many big words in there.

Samantha was explaining that once you figure out the number of seats around 10 tables, you can be adding 20 seats every time 10 tables are added to the train. Most students reported that this strategy was difficult to understand and it was among the last presentations of the first day's discussions about rules for finding number of people around a train of 100 square tables. In addition to sequencing student responses according to levels of sophistication, the teacher considered responses that were expressed differently or were contradictory to be presented one after the other. The teacher explained in the following interview excerpt that such sequencing created more opportunities to ask for justification of responses. According to the teacher, sequencing responses according to the levels of sophistication and according to the differences in the reasoning was purposefully done to include more strategies and solutions for public display while simultaneously staging the students to notice how the strategies and tasks were related to each other. This approach supported mathematical argumentation, conceptual understanding, and co-construction of ideas.

> *Teacher*: Well, you always hope that there are two different views or two different ways of thinking about the task. Half the class thought one way (202 people can sit around a train of 100 square tables) because they used one strategy to get the answer. The other half used another strategy (and found 220 as number of people who can sit around a train of 100 square tables)…. You accommodate the differences when you do large group instruction and when you have different ways of doing it. So you keep on stressing, does somebody have a different way of doing this, did somebody get a different answer? And you set it up so that you can say, ok, let us see if we can justify both of these answers, may be they are both correct but you (students) have to justify your answer (interview).

4.5 Connecting

The teacher explained that she attempted to help students make several connections. The connections she made included connecting responses across different tasks, connecting different students' responses, and connecting justifications and generalizations.

Connecting different student ideas. Whole class discussions at each critical phase of the tasks (modelling and data collection, generalizing recursive rules, generalising and justifying explicit rules) afforded students opportunities to connect their reasoning to other students' reasoning. Similarly, the teacher constantly encouraged students to evaluate each other's argument as in this instance:

> *Brenda*: Multiply the tables times two and then add two to find the number of chairs.
>
> *Teacher:* [to the whole class] Think about that (rule) one more time, does that work?
> *Stan*: Yeah.

Teacher: How do you know it works?

King: Because T (input) times two every time is two less than C (output) and if you add two plus T it works.

Teacher: And Sam, what did you say?

Sam: It does work.

This is an instance of classroom interaction from the first day of the teaching experiment as students were working on task 1, in which the teacher encouraged students to reason as to why their classmates' rules were valid or not. In this instance, Brenda presented a rule for finding number of seats available on a train of n square tables and the other students evaluated why that rule could be valid. It was observed that this approach encouraged students to incorporate other students' strategies into their own. For example, after one student used a geometric representation to justify the rule $2n+2 = p$ for a train of square tables, most students incorporated that representation into their explanations. This encouraged the students to evaluate others' ideas and connect peers' arguments to their own reasoning. Another event on the second day was when students were justifying their rules. Becky explained that the rule $n+2 = p$ was valid for a train of square tables because every time 2 tables were being put together, the train was losing 2 sides that were touching, which were previously available. The following argumentation and thinking (classroom interaction during the first day) followed. This argumentation based on students' perceptual schemes went on, bringing in new understandings. The teacher finally asked Benny, who had a different justification. He explained that the rule $t+2=p$ holds because "the plus 2 comes from the end sides of the table (train)" and the number of people is constantly equal to the number of tables in that pattern.

Sam: … speaking of the way she said, you are not subtracting two, you are adding two.

Teacher: She said that you are (your rule is) not taking away, you are (it is) adding two to the number of blocks.

Connecting ideas across tasks. Several attempts were made by the teacher to connect students' reasoning across tasks. The instructional tasks were isomorphic and she asked questions that encouraged students to reflect on previously worked out tasks to reason about new tasks. For example, the teacher mentioned the square table trains while the students worked on triangle table trains, and one student said, "I don't know where we got two hundred and twenty. You would have to have a hundred and twenty tables—a hundred and ten." This student reflected on his faulty reasoning about the number of people who can sit around a train of 100 square tables. With that reflection, he tried to figure out what the question would have been for his response to be correct. Similarly, the teacher probed the students to think about previous justifications as they worked on trains of hexagon tables. This probe was followed by Jane's analytical thinking:

Teacher: Remember on the first day when we did squares (task 1)? …(Use it) to think about the hexagons.

Jane: So, a hexagon has two sides, so it has two hundred on each side. And two hundred times two is four hundred plus the two on the ends is four hundred and two.

The teacher explained that she attempted to connect ideas across tasks to position students to see the relationship between tasks and scaffold their understanding. She further explained that although with these tasks she supported students' understanding by connecting to reasoning in isomorphic tasks, connecting to their other everyday activities might also scaffold students' understanding. She said:

Teacher: It is reminding them that perhaps there is a relationship between what we did yesterday and what we did today. That we are going to build trains of tables but with a different shape. So, it is kind of saying, what we did yesterday was important. Now today we are doing something like it only a little different….Now, it just so happened that the connection was from the previous day. You could have just as easily said, remember (what we did) two weeks ago (interview).

Connecting recursive and explicit rules. The task set-up positioned students to make recursive rules first and then explicit rules (see figure 1). As the teacher explained:

Well, you want them to see the recursive rule first. And so in building the table they see the recursive rule. That is important. Also building the table, you hope, leads to focus them on both variables. It does not always happen, because they usually begin focusing on one variable, but you hope that eventually they will go to both.

She set the tasks this way so that students could make both recursive and explicit rules. During the lesson, she encouraged the students to develop explicit rules by asking them to figure out how many people can sit around a train of 100 tables without using a brute force approach. Throughout the teaching experiment, she asked, "how would you figure out how many people could sit around 100 of these tables that were all joined together? Without figuring out all the ones by adding two, what is a quick way of doing that?" (Class interaction on days 1, 2, and 3)

Connecting responses to authorship. Class discussions showed that the teacher communicated to the students that they owned their strategies and solutions. One of the ways the teacher did this was by constantly using "your rule" or "your strategy" as opposed to "a" or "the" rule. She also empowered students to validate their knowledge through mathematical argumentation rather than looking up to her as the only authority as in this classroom interaction from first day.

Logan: What is the (correct) answer?
Teacher: Do you believe in your answer (202 for a train of 100 square tables)?
Logan: Yeah.
Teacher: You do? This table thinks it is 220. Could you prove to them that you are right and they are wrong?

Consequently, the students showed ownership of their ideas by referring to the rules and justifications as either "my rule" or "our rule" or strategy. The teacher explained that

It is important for them to own their answer, to be proud of their answers, and to be confident in their answers. So rather than to say it is right or wrong, I wanted the children to be able to figure it out among themselves and then use probing questions to move them along (interview).

Connecting the lesson to unanticipated student ideas. The teacher took up unanticipated ideas or questions to enhance students understanding. For example, Dan modified the set of tasks given by the teacher and asked how many people would sit around a train of 50 square tables (day 1). She posed Dan's question to other students.

> *Teacher*: Dan had an interesting question that he posed to me. He said, "Would two groups of 50 tables give you the same answer as one group of 100 tables?"
>
> *Student*: No. Because there would be four extra. Because if you had 50 going this way (a train of 50 tables) there'd be one and one (one seat on each end of the train).
>
> *Dan*: There would be two groups of 50.
>
> *Student*: It would be 204. Thank you, Dan.

The teacher explained that this was "the case where students' ideas are sometimes much more powerful than the teacher's. So it was even more important for me to depend on students who can put up an idea that all the others will adopt." Thus, highlighting powerful unanticipated ideas was important.

Connecting verbal and symbolic rules. The teacher also made attempts to position students to connect verbal and symbolic rules. For example, when a student presented her symbolic rule in figure 2, the teacher asked: "So what would that look like in words?" The student then explained her rule in words.

Figure 2: Rule for square tables.

Connections between verbal and symbolic rules were made so that as many students as possible could understand the symbolic rules as the following interview response suggests.

> I knew that most children did not understand her (symbolic) notation. So I wanted her to interpret her notation in words for those students who had an explicit rule but would think their rule is different from hers when actually it was the same rule.

Connecting reasoning to task's context. When the students expressed their explicit rules, they were asked to explain how their rules related to the context of the problem. As an example, the following class interaction was on the third day when a student expressed $4t + 2 = p$ as a rule for finding the number of people that can sit around a train of t hexagon tables.

> *Teacher*: Okay. So do you know what the two is?
>
> *Bea*: The two end chairs.
>
> *Teacher*: Yeah?
>
> *Bea*: Like if they're like this (models the two ends of the hexagon train).
>
> *Teacher*: Yeah, and where does the four come from?
>
> *Bea*: From– you have times two (seats) up here (at the top of the hexagon table model) and times two (seats) down here (bottom of hexagon table model), so that's four times…

5. Summary and Discussion of Results

The results of this study are summarized in Table 4. Although these results are specific to algebraic thinking in elementary classrooms, they are in concert with findings from Stein et al.'s (2008) literature review. Anticipating calls for content knowledge for teaching mathematical reasoning, which includes knowledge about reasoning tasks and the type of reasoning that such tasks can support (Stylianides & Ball, 2008). Working out the tasks —especially with other teachers— and keeping a record of practice (e.g., student worksheets and records of classroom related scenarios) can broaden a range of anticipated strategies.

Monitoring positions teachers to select responses purposefully rather than randomly choose presenters for whole group discussions. For equitable classrooms, selecting responses from many students and assessing patterns of selection over time is recommended. Monitoring patterns of selection may require teachers to keep a record of presenters for each day. Purposeful selection may contribute to building norms that reasoning tasks may be solved using multiple acceptable strategies —a practice that is key to fostering reasoning (Rathouz, 2009) which many teachers find challenging (Depaepe, De Corte & Verschaffel, 2007). In addition to sequencing selected responses from less sophisticated to more sophisticated, contradictory responses create opportunities for students to engage in justifications and refine their reasoning (Komatsu, 2010). Students who justify or evaluate such ideas are afforded mathematical expertise authority (Gerson & Bateman, 2010). That is, when students have a sense of authority that comes by authoring or co-authoring an idea and authority that comes by evaluating or justifying ideas, their potential for mathematical understanding and mathematical autonomy is highly supported. Such mathematical autonomy may nurture progress from using external justification schemes to using contextual justifications. According to Greeno (2006), contextual justifications are a tool for transferring knowledge from one context to another.

Sequencing can be a tool for connecting ideas and for conceptual understanding. Conceptual understanding can also be supported by making student ideas focal during whole class discussions (Piccolo, Harbaugh, Carter, Capraro & Capraro, 2008). Prior research reported that isomorphic tasks support reasoning because the structure of the tasks supports transfer of mathematical ideas (Richardson, Berenson & Staley, 2009). Connections can also be across tasks that are not necessarily isomorphic. This fosters an understanding that mathematical concepts are not distinct but connected, in that an understanding in one area might foster an application in another area. Connecting word and symbolic generalizations could improve students' understanding of symbolic generalizations, which is a challenge to many students (Capraro & Jofrion, 2006).

Table 4

A summary of Discursive Practice for Supporting Algebraic Thinking

Broad category	Subcategories of practices for supporting algebraic thinking from these study's data.
Anticipating	Anticipate 1. Students' tendency to make recursive rules. 2. Limited mathematical language pertinent to algebraic thinking. 3. Patterns that can be easily linked to their geometric representations to be easily noticeable and justified. 4. Multiple strategies including a focus on • Varying and constant aspects of patterns. • Number of seats lost by joining the tables into a train relative to number of seats that would otherwise be available if the tables were disjoint. • Number of seats at the top, bottom, and ends of the train. • Number of seats each of the tables on the train is contributing.
Monitoring	Interact with the students during small group discussion to 1. See and hear their thinking. 2. Collect data to inform the teacher on how to progress with the lesson. 3. Identify ideas that should be made focal during whole group discussion. 4. Formatively assess students' reasoning.
Purposeful selection	Criteria for selecting responses for public display include 1. Strategies that appear different. 2. Recursive and explicit rules with different sophistication levels. 3. Surprising responses. 4. Different students to make participation equitable. 5. How well the student responses connect with objectives.
Purposeful sequencing	Responses may be purposefully sequenced by 1. Having multiple small group and whole class discussions in one lesson. 2. Moving from less sophisticated strategies (e.g. recursive rules) to more sophisticated strategies (e.g., explicit rules). 3. Moving from less to more accessible ideas.
Connecting responses	Explicit mathematical connections should be made between 1. Different student ideas. 2. Ideas across different mathematical tasks. 3. Recursive and explicit mathematical rules. 4. Verbal and symbolic rules. 5. Responses and their authorship.

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Monitoring is essential in supporting students' reasoning as students work independently or in small groups. In this study, the teacher discussed that it is a challenge to help students negotiate their mathematical meanings while they work in small groups. That is, although students are supposedly working as a group, they work independently and share or report their thinking to their partners. Further research may look into teaching practices that support students' co-construction of mathematical ideas while working in small groups. More studies on classroom discourse should be conducted as classroom discourse is "pivotal to current reforms in mathematics education because discourse informs not only our understanding of students' thinking about mathematics, but also teachers' thinking about teaching mathematics" (Blanton, Berenson, & Norwood, 2001, p. 227).

6. Implications

Good's (2010) review of the literature revealed the persistence of univocal discourse (Wertsch, 1991) in math classrooms with the teacher communicating ideas and their meanings to students for students to accept those ideas as they are. The emergence of the NCTM's Standards and Principles, and CCSSM's practice standards is a reflection of how evidence has steadily supported reformed practices —a shift from univocal to dialogic discourse in which dialogue is a thinking device and meanings are co-constructed between teachers and students and between students. Truxaw and DeFranco's (2008) analyses of middle school teaching urged our view of classroom discourse to move beyond merely considering the types but rather the quality in relation to how it supports student learning; arguing univocal and dialogic discourse are on a continuum and both serve essential roles in the classrooms. While we agree with Truxaw and DeFranco, we believe high-quality dialogic discourse is at the heart of the nature and maintenance of cognitive demands of generalisation and justification tasks. The intersection of the persistence of univocal discourse over time and space, and the affordances of discursive practices that emerged in this study present a fertile field to ask: How can teacher education and development support a shift towards productive discourse?

A few studies have explored this question. Blanton (2002) found that learning experiences in math content courses are opportunities for teachers to reflect on discourse and frame the desired discursive practices for their future classrooms. When prospective teachers experience high-quality dialogic discourse, they are likely to normalize such discourse and embrace reformed teaching practices that support generalisations and justification mathematical practices. Drawing from Ball and Forzani (2009), perhaps another obvious approach is for teacher education to foster more intentional teaching practice of high-quality dialogic discourse. From many studies on mathematics professional development (e.g., Garet, Porter, Desimone, et al., 2009), analysing teaching videos to notice high-quality discourse would also be productive. Further, Marrongelle, Sztajn, and Smith (2013) insisted that "we need studies that open the black box of PD and provide rich descriptions of the nature of the work in which teachers engage that does or does not lead to improved knowledge, beliefs, or habits of practice" (p.209).

7. Essence of this study

As discussed in the methodology section, the teaching experiment was not designed to be a confirmatory study for the Stein's framework. The choice of the framework was a post hoc and naturally emerged from line by line coding as the framework that explained most of the data. Since the Stein et al.'s (2008) framework was used theoretical class scenarios and did not study all these practices as connected pieces, although the data analysis framework was a post hoc, this study confirms how productive the discursive practices are. Additionally, the Stein et al.'s framework is very general, it does not fully provide frameworks for different mathematical domains. To draw richly on the affordance of dialogic discourse, teachers and teacher educators need practical frameworks, and these will naturally be different between mathematical domains.

For example, good teaching practices realise the productive connections when teaching the Pythagorean Theorem will be different from those of a statistics unit on Sampling or Algebraic Thinking. This study contributes to Stein's framework by providing a mini and practical framework for algebraic thinking.

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