IS MATHEMATICAL KNOWLEDGE *CONSTRUCTED*? A CULTURAL-HISTORICAL CRITIQUE OF OBJECT ORIENTED CONCEPTIONS OF LEARNING ACTIVITY

O CONHECIMENTO MATEMÁTICO PODE SER CONSTRUÍDO? UMA CRÍTICA HISTÓRICO-CULTURAL ÀS CONCEPÇÕES DE ATIVIDADE DE APRENDIZAGEM ORIENTADAS AO OBJETO

Wolff-Michael Roth
mroth@uvic.ca

University of Victoria

ABSTRACT

It has become a truism (ideology) to state that mathematical knowledge is *constructed* collectively, in communities of practice, and individually, on the part of students while engaging in mathematical tasks. However, *construction* implies an image of the end result of the labor process, which allows people to build a house and compare each step to the plan. Students, on the other hand, do not know the end product of their learning process, the *new* knowledge. This knowledge, therefore, cannot be the transitive *object* towards which construction is oriented. In this study, I provide a cultural-historical critique of object-oriented notions of learning activity. Using classroom episodes as examples, I propose an alternative based on L. S. Vygotsky's commitment to the primacy of the social, whereby any higher psychological function *was* a social relation first. This allows the final product to be available in the present, as relation, without the learner's conscious awareness, and thereby determine learning and development. The idea of the future acting in the present is captured in M. Cole's notion of *prolepsis*. Implications are discussed with respect to curriculum design in mathematics classrooms.

Keywords: Activity Theory; Contradiction; Learning Paradox; Leading Activity; Mediation; Zone of Proximal Development; Prolepsis

RESUMO

Tornou-se um truísmo (ideologia) afirmar que o conhecimento matemático é construído coletivamente, em comunidades de prática e, individualmente, por parte dos alunos, enquanto se envolvem em tarefas matemáticas. No entanto, construção implica em uma imagem do resultado final do processo de trabalho, a qual permite que as pessoas construam uma casa e compararem cada passo do plano. Os estudantes, por outro lado, não conhecem o produto final do seu processo de aprendizagem, o novo conhecimento. Este conhecimento, por conseguinte, não pode ser o objeto transitório para qual a construção é orientada. Neste estudo, eu forneço uma crítica histórico-cultural de noções da atividade de aprendizagem orientadas ao objeto. Usando episódios de sala de aula como exemplos, proponho uma alternativa baseada no compromisso de L. S. Vygotsky à primazia do social, segundo o qual qualquer função psicológica superior foi uma relação social em primeiro lugar. Isso permite que o produto final esteja disponível no presente, como relação, sem a consciência do estudante, e assim determina a aprendizagem e o desenvolvimento. A ideia do futuro agindo

no presente é capturada na noção de *prolepsis* de M. Cole. Implicações são discutidas em relação às propostas curriculares para as aulas de matemática.

Palavras-chave: Teoria da atividade; contradição; paradoxo de aprendizagem; atividade principal; mediação; zona de desenvolvimento proximal; prolepsis.

1. Introducing the problem

> "A spider conducts operations that resemble those of the weaver, and a bee puts to shame many a builder in the construction of its wax cells. But what distinguishes from the very beginning the worst human builder from the best bee is that he has built the cell in his head before building it in wax. At the end of the labor process is a result that was already there in the beginning ideally, in the imagination of the worker. He not only brings about a change of form in nature; he simultaneously realizes in nature a purpose [motive], which he knows, which determines the manner of his doing as law and to which he has to subject his will." (Marx & Engels, 1962, p. 193)

The human work has a goal. The worker orients towards the image of this goal, as Marx and Engels suggest in the introductory quotation, in such a way that the materials at hand are transformed until they will have taken the form corresponding to the image. In so doing, the worker achieves two things: bringing about change in nature in the form of an outcome produced and the materialization of a motive (idea become thing). That materialization requires thought to submit to the material conditions. Interesting about the quotation is that Vygotsky (1997a) uses it to open a chapter on consciousness as a problem for the psychology of behavior. It is in consciousness that the future state of activity, its product, exists as the motive. That is, at the beginning of the productive process, its end already exists in consciousness together with the materials to be transformed. Unsurprisingly, perhaps, cultural-historical scholars frequently write about the *object/motive* of activity. Numerous studies of mathematical learning theorize their studies in terms of object-oriented activity (Kieran, 2002; van Oers, 2001) or explicitly draw on cultural-historical activity theory (e.g., Beswick, Watson, & de Geest, 2010; Jaworski & Potari, 2009). This theory is epitomized in a triangular representation that makes stand out different levels and degrees of mediation (Figure 1) (Roth & Lee, 2007). The sense of current activity arises from and exists in the objective relation between what is at hand, the current material situation, and the result of the activity to which it is oriented (Leont'ev, 1982).

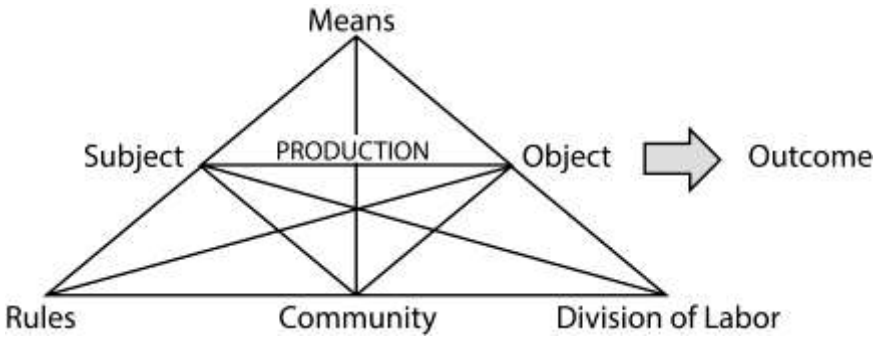

Figure 1. A cultural-historical activity theoretic approach explicitly thematizes production, from object to outcome, which is anticipated and exists as image from the beginning.

In other theories, researchers also at least imply an object; but most theories in education are about mind and its constructions, not about mind as a manifestation of a higher unit that also manifests itself in practical labor. The most elaborated of such theories derives from phenomenology, where any *human* engagement with a material thing involves sensual experience and consciousness; this double relation of the person to the object characterizes intentionality (e.g., Husserl, 1973). Vygotsky (1997b), following Spinoza and others, uses the case of (a) Buridan's ass, who, unable to decide between either of two food sources at equal distance, starves to death and (b) the human being, who, because of consciousness, is not subject to a determined response in the situation. That is, if someone or a group of people are said to "construct" *something*, there are always the things that enter the constructive work and the appearance in consciousness of some outcome.

Whereas the foregoing description of human activity is correct in the case of the builder, and all cases that have similar structure, it is inappropriate as the basis for theorizing *learning* activity. As noted, the *sense* of an activity arises from the objective relation between what is, now, and the result of the activity. But for the learner, this result does not exist practically and cannot yet exist ideally. This situation has been articulated in the case of the "poet," a person who creates something new, who is "typically unable to make clear exactly what it is that he wants to do before developing the language in which he succeeds in doing it" (Rorty, 1989, p. 13). It is only when the creator has had insight into the new that she can also describe what she has been doing along the way. The insight is required to do "something which could not have been envisaged prior to the development of a particular set of description, those which [the new language] itself helps to provide" (p. 13). Consciousness, for Vygotsky (1998), is relation to the environment. But for human beings, this is not a simple relation. He makes reference to Marx, who states, "the animal '*relates*' to nothing and not at all and in general does not 'relate'; for an animal, its relation to others does not exist as a relation" (Marx & Engels, 1978, p. 30). This is so because for the animal the perception of the surrounding world and its experience are fused, whereas for the human being next to perception there also is consciousness—Marx notes in the same book that consciousness [Bewusstsein] is conscious being [bewusstes Sein]—that is irreducible to internal or external perception.

This study was designed to problematize the idea that mathematical learning activity can be intentionally oriented to its outcome as the object. That is, precisely because learners do not already know what they will know in the future (object), they cannot intentionally construct this future knowledge. Following the statement of the theoretical position taken here, which is the Spinozist orientation that Vygotsky was taking towards the end of his life, I present a case study that illustrates the learning paradox and its resolution in one (successful) situation. In discussing the case study, I suggest that the idea of *mediation* leads to contradictions; and it is inconsistent with the Spinozist turn Vygotsky was taking. I end with some ideas about addressing the learning paradox in teaching practice.

2. Theoretical background

From a cultural-historical (dialectical) perspective, psychological categories are plausible only if one can show how the associated phenomena have emerged in the course of human evolution (e.g., Holzkamp, 1983; Il'enkov, 1977; Vygotsky, 1997a). From a psychological perspective, therefore, we have to ask when the voluntary goal in the life activity of our forebears appeared, and, therefore, when the possibility of an image of the goal first emerged (Mikhailov, 2004); and we have to question the possibility for the image of the goal of

activity in ontogenesis of individuals that attend our mathematics classrooms and engage in the curricula we design. The emergence of this image, its very possibility, has to be shown and explicated as a *new* psychological formation, a new quality or qualitatively different phenomenon. Although it is qualitatively different, it does arise from and has its seed in current activity. However, that seed cannot be the result of a mere quantitative-continuous change: we need to account for the emergence of a qualitatively *new* formation. "From a purely logical point of view, it is necessary to derive the *new* not as something that has already existed inside the *old*, but as something that *negates* this old" (Mikhailov, 2004, p. 15).

In the majority of psychological approaches, including those that claim to be activity and enactivist approaches in psychology, the specter of Cartesian dualism (parallelism) looms (Mikhailov, 2004; Sheets-Johnstone, 2009). In fact, constructivism, with its singular orientation on the intellect and agency, inherently fails to capture much of human experience, including affect and passibility, and, therefore, constitutes a theory full of contradictions (Roth, 2011).[1] The very idea of human activity as object-oriented activity that is—in contrast to the animal's direct relation to the world—*mediated* by tools, rules, community (culture), and division of labor implies a parallelism of nature and culture that needs to be bridged by means of mediators so that explanations of life actually involve both (Mikhailov, 2004). This, then, "leads to something in direct opposition to Vygotsky, for whom the fact of different manifestations, natural and cultural, of the process of emergence of man's psyche is the realization of a *single* foundation" (p. 19). That single substance (i.e., self-creating nature), following the philosopher B. Spinoza, has two attributes, two ways of manifesting itself: thinking (mind, consciousness) and corporeality (body). That is, thought is a manifestation of nature transforming itself: "Consciousness determines life (the image), but it derives itself from life, and forms its component: ergo life determines life [itself] through consciousness" (Vygotsky, in Zavershneva, 2010b, p. 48).

In the Spinozist take that Vygotsky was taking towards the end of his life, the "voluntary and goal-conforming life-activity is its own cause" (Mikhailov, 2004, p. 25). *New* psychological forms are generated in practical activity, which, in humans, always also is intellectual and affective. This cause cannot be a mediator, which would stand between nature and the individual, "because it is precisely the mediator that predetermines the contrary dichotomy and eternally inescapable counterposition and opposition between man's subjectivity and the objectivity of nature" (p. 25). Moreover, these new psychological forms, because they are qualitatively different from the present, cannot be mere extensions of already existing knowing-how. The problem of activity as its own cause (*causa sui*) is solved when subjective motivation coincides with whatever objects that subjectivities are oriented to. This occurs precisely in the "mutual relation whose object is precisely the externalized subjectivity of the other" (p. 25). Confronted with the externalized subjectivity of the other, the intellect "is addressed not to dead, unthinking, and unfeeling nature (an object) but to the affective-intellectual receptivity of another human beings—and always, therefore, to our self as its possessor" (p. 26). The result of this affective-intellectual receptivity is that the human being becomes the subject of its own voluntary and goal-confirming activity.

3. An episode from a fourth-grade mathematics lesson

[1] In constructivism, the material reality is fundamentally alien to the knowing mind; the mind only and somehow tests the viability of its own constructions.

In this paper, I draw on an episode from a fourth-grade mathematics classroom doing a special curriculum on pre-algebra that nevertheless fit into the guidelines of the Canadian province where the events took place (Roth & Radford, 2011). The task describes the situation of a girl receiving a piggybank containing $6 and deciding to save $3 each week. In the first part of the task, the students are asked to use goblets and colored chips to model the situation for each of six weeks. In the second part, they are to fill the following table of values:

Number of the week	**1**	**2**	**3**	**4**	**5**	**6**
Amount saved ($)	+6	+6				
Or	+6	2x +6	3x +6	4x +6	5x 6	x +

Mathematically knowledgeable readers will recognize that the second row may be seen as calling for the $3 amounts saved each week, amounts that need to be thought to be *in addition to* what was already in the piggybank. But there is also a potential ambiguity, as the heading calls for "amount saved ($)" rather than for the total amount in the piggybank. As shown here, yellow highlighting was used to make the identity of the number of the week and the multiplier stand out. The cognitive complexity of the task exists in the fact that for each week, the contents of the piggybank need to be thought iteratively, so that each cell requires restating the previous week to which the $3 amount is added. The third row then calls for converting the repeated additions from the cell above into a multiplication. The goal of the curriculum is for students to arrive at something equivalent to the generalizing statement, "the number of the week times $3 plus the $6 dollars we started with."

In the account below, we follow the camera to an arrangement of four student desks, where there are three students (Aurélie, Mario, and Thérèse), who were asked to work as a group but who de facto work on their own.

3.1. The contradiction

The goal of the task, as per the teachers and curriculum, was for students to arrive at a generalization about the piggybank contents. The contradiction that the students are faced with pertains to the fact that they do not know *that*. They do understand that they are to model something using the goblets and colored chips; and they do understand that they are to fill up the table. But filling up the table cannot provide direction as to what to fill it up with, just as saying, "Fill the cup!" does not tell the recipient of the command *what* to fill the cup with. The students cannot know the end result of what they are doing, and, therefore, cannot intentionally orient towards it; the product—i.e., the generalization—cannot already exist in their individual or collective minds in the way that Marx, Vygotsky, and activity theory stipulate it as the characteristic of productive activity. Moreover, if the outcome to be produced is unknown, it is impossible to assess whether one is on the right track—in the way the builder can assess whether what has been done so far corresponds to the architectural plan that anticipates the final building. The assessment where someone is with respect to the anticipated outcome is reflected not only intellectually but also affectively. If the outcome is unknown, a person might sense being lost or may feel not knowing what s/he is doing. The fragment exhibits both intellectual and affective manifestations of that contradiction in the case of Mario.

The fragment takes us into the part of the episode where the teacher (Jeanne), being called by Mario's raised hand, has joined the group where she works above all with the young lad.

After filling the goblets, Mario has tried for a while filling the table; he has looked at what Thérèse was doing, but notes, "I don't know what you are doing." He eventually raises his hand, waiting for the teacher to join them at the group of desk. Whereas Thérèse appears to fill up her table, Amélie already has given up, providing many manifestations of frustration and other forms of negative affect (e.g., Roth & Walshaw, 2015). We are entering the exchange between Mario and Jeanne when she holds up the goblet marked and offers a query, "How much money is there already in the third week" (turn 150).[2] There is a long pause (i.e., 3.42 seconds)—long given the fact that the standard maximum silence is about 1 second and given that teachers tend to leave only about 0.8 seconds to respond before continuing to talk—then a repetition of the query. Again, there is a pause, interrupted by some interjections (turns 151–154), before a first reply ("fifteen") ascertains the querying function of turn 150. A second reply follows, produced with much lower than normal volume, "twelve" (turn 157). The teacher has the next turn (turn 158). That turn repeats the words from turn 150, but does so following turn 157, thereby questioning and negatively evaluating the latter. In relation to turn 160, which repeats the previous reply, turn 158 has the function of a question. Turn 162 treats this reply as problematic in not only asking for an account ("Why?") but also for the composition of "it." The take-up of this turn manifests itself intellectually and affectively, expressions that we may gloss by "look, what do you want me to do with that?" while pointing to the table of values, with both hands, palms turned up (turns 163–165)

Fragment 1

150 J: its t*wel*ve ((*confirming, nods deeply, open right-handed gesture, palm upward*)) (0.94) so (.) how much money is there in, how much money (0.92) (2.50) how m*uch* money is *the*re (.) *in*, (0.29) already in the third week ((*raises goblet, jingles it, places it back*))
151 (0.79)
152 T: um um u::m.
153 (1.12)
154 M: u:[m::
155 T: [fifteen]
156 (0.30)
157 M: °t*wel*ve.°
158 J: =how m*uch* should th*e*re already be.

[2] The following transcription conventions are used. All text is in small letters; ((*confirming*))—transcriber's descriptions; (0.79)—pause in seconds; [m:::]/[fifteen]—brackets indicate overlapping speech; :—colon indicates lengthening of sound by 0.1 second per colon; °twelve°—degree sign mark considerably lower volume; d*oll*ars—italics indicate intonational stress; .;,?—punctuation is used to indicate intonational lines in statement, strongly falling, falling, rising, strongly rising, respectively; (.)—noticeable pause shorter than 0.1 second; i=think—equal sign indicates latching; and six dollars—grey shading marks overlap of image with speech. Images have been produced from raw video using Photoshop "Dry Brush" filter.

159 T: u:h:
160 M: twelve
161 (0.21)
162 J: w*hy*. *it*s composed of what.
163 (0.68)
→ 164 M: what well l*ook* ((*frustrated, hands stretched out, palm up, toward worksheet*))
165 (0.27)

166 J: twelve d*oll*ars cont*ain*s the::?
(1.48)
six d*oll*ars that we start w*ith*?(0.46) and how m*uch* money in the other two weeks be*fore*
167 (2.01)

168 M: what?
(1.56)
that makes-
(0.80)
→ i dont unders*tan*d (.) th*ou*gh.
169 J: °you dont understand that° its what i=m trying to help you understand
(2.40)
l*ook* well
(3.50)
are we l*ook*ing (0.65) trèse?

Responding to these manifestations, Jeanne then provides what we can hear to be the beginning of a reply to her own question, the $12 are composed of the six dollars that they started out with, which leads into the offer of an invitation to provide whatever the slot following "and" calls for: $6 and __? There is a long pause (2.01 seconds), an interrogative ("What?"), and another long pause (1.56 seconds), the beginning of a reply, and another pause. We then hear an intellectual expression of lack of understanding, intonated in a way that manifests frustration; and we see Mario physically manifests that frustration as well when he props up his head by means of the two hands, gazing toward the table of values (turn 168). That description is taken up in an acknowledging manner in the next turn. There is then what technically is a *formulation* or *gloss* when Jeanne states, "it's what I'm trying to help you understand" (turn 169). That is, this is an insider formulation of how the fragment has to be read, as an account of the work that they were doing in the course of the events represented in Fragment 1.

The fragment manifests the practical contradiction that exists for students, here exemplified by Mario, who are to orient towards the object of learning and "to construct" (here algebraic) knowledge. We may gloss the situation in this way: Mario does not know what to produce, and, therefore, cannot have any inkling about *how* to produce. He cannot even conceive of

the source of the problem, and, as suggested in turn 169, may not be able to understand that the teacher is trying to help. Mario does make available his lack of understanding, here pertaining to what and how to put into the cells of the second row in the table of values. The situation also manifests itself affectively in ways that culturally savvy individuals can see to be frustration.

Here, then, a second contradiction offers itself. Mario is frustrated. Indeed, he has increasingly become frustrated, and repeatedly expressed here and before not understanding. He is conscious of being lost, and this consciousness, in and through his speech, is manifested objectively in the situation, allowing others to be conscious of this fact as well. Moreover, as shown in the continuation of the events, as Mario becomes even more frustrated, Jeanne also manifests frustration. She is trying, as articulated in turn 169, to help Mario understand, but, as shown in a subsequent statement, fails to do so. The contradiction exists affectively in that the only chance to overcome frustration is to continue working. Even though both experience frustration, the only hope to get out is to continue. They have to do so without the guarantee that they actually overcome frustration to reach a state where some success has been achieved, which then could be valuated positively in affective terms. Elsewhere, we describe the situation from the perspective of another girl in the group, Aurélie, who abandons the task and eventually copies what the third student, Thérèse, has on her worksheet (Roth & Walshaw, 2015). Aurélie never did get out of her frustrations, manifested among others by her pounding of the desk with the fist and by placing the head on the desk as if she were sleeping. That is, the only hope Mario and Jeanne have to change the situation is to engage in joint work, colored through and through by negative emotions, to bring about an intellectual (i.e., Mario's understanding) and affective turn (i.e., positive emotions).

This also means that Jeanne has to find a way that initiates the turn-around. Up to now she has not succeeded. Thus, apparently, she has not acted in a way that would allow Mario to understand. In fact, in the exchange, she has to find out, and therefore learn, how to assist a student such as Mario to come to comprehend what the task is asking him to do. Therefore, both have to learn as they go on, the difference being that Jeanne already knows what Mario is to do and when he does something that does not lead to the stated curriculum outcome. She has to find (learn) a way to get out of the situation. Her frustration is associated with not having been able to assist Mario to understand how the goblets and what he has been doing with them is to be translated into the contents of the second row cells.

3.2. The future in the present

Mario will be able to understand what he is doing and why only by doing it. He can do so once he has arrived where the curriculum is to take him. This work that takes him there cannot be foreshortened by telling him the answer. We see that it does not help in the cause of Aurélie, who copies the results that Thérèse has entered. Aurélie continues to be frustrated and, in the end, does not understand even though the results and the way to get there are described. Much like a cooking recipe, which we do not understand until we actually have achieved doing the work of which it is an account, *knowing* the content of such generalizations as "number of weeks times three dollars saved plus the six dollars you started out with" requires having done the work of generalizing. That is, the practice of generalizing exists in the combination of the work of generalizing plus an account. The structure of this practice, therefore, may be denoted by "doing [generalizing]"; and knowing this practice means being able to do the work designated by "doing" in its relation to the account

designated by the contents of the square brackets (i.e., [generalizing]). In the cultural-historical approach of Vygotsky (e.g., 1989), the work itself is social and the joining of the two parts, the doing and its account, will have been a social relation first (e.g., Roth, 2016). As such, the practice already exists *before* Mario actually acts it out on his own, as he does subsequently. This phenomenon has been termed *prolepsis* (Cole, 1996), the existence in the present of what later will be a higher psychological function. In Fragment 2, we glimpse this phenomenon.

With Fragment 2, we step into the event after something similar has already occurred for a first time, and which now leads to the evaluating statement that manifests Jeanne's sense that Mario now understands (turn 227). We observe how the two table cells corresponding to week 4 come to be filled, as manifested in the protocol of the work represented by Fragment 2. First, there is a query | reply sequence, "how many three dollars do you have | fo[ur]" (turn 28 | turn 220) that are separated by a long pause. This pause is itself interrupted by an interjection "u:m:::" that has the function of marking that Mario has taken the floor but that he is not yet ready to produce the reply. In reply to Mario's entering the four "3"s in the cell of the second row, another query is offered, formulating what is to come in the reply as something that is instead of "doing three plus three plus three plus three" (turn 222). In the same pattern as before, there is a long pause interrupted by an interjection before the query-constituting reply is offered, "four times three?" (turn 224). But, as the question mark shows, the statement is offered not just as a constative, factual statement, but also as a question (i.e., intonation rises towards the end). The statement is offered as a question, as if he had said, "Is 'four times three' the right answer?" There first is an affectively charged hand gesture, which we easily see as manifesting something like "You got it." That reply is followed by a statement (which also is a question, as the intonation rises towards the end): "I think you understand now. Uh?" (turn 227). Mario continues filling up the four cells below week 5 and week 6, which takes him about 51 seconds. He then states in what can be heard as a confident manner, "Me, I understand now" (turn 229).

Fragment 2

	218	J:	your fourth week; (.) how m*an*y three dollars do you have.
	219		(1.00)
	220	M:	u:m:::
			(1.73)
→			fo. ((*Fills table cell.*))
	221		(9.48) ((*writes four times a '3'*))
	222	J:	°°kay°° (0.97) instead of doing three plus three plus three plus thr*ee* wh*at* are you doing to wr*ite* here? ((*Points to the row on the bottom of the table of values*))

223 (0.66)
224 M: uh:m::
(1.36)
four times thr*ee*?
225 J: ((*2-handed gesture "you got it"*))
226 (3.83)
227 i=think you understand now. uh?
228 (50.93) ((*Mario slightly nods, writes, after 26 seconds looks at Therese's worksheet, back at his own*))
229 M: ((*confident*)) *me* i understand now.

After the teacher statement concerning Mario's understanding, Mario fills up four table cells. It is apparent that he now is doing on his own what before that he participated in doing together with Jeanne. That is, links come to be established through their joint work between Mario's earlier actions of filling the goblets with chips and the corresponding accounts. Thus, for example, he had taken the goblet marked with the number 4, filled it with the amount that was already in goblet 3, and then adds 3 more chips. In turn 220, he says "fo[ur]" and fills the second-row cell writing "3+3+3+3+6." This written statement is the (verbal, mathematical) *account* that goes with the work of putting the corresponding number of chips. The link between the two, however, is not seen in the behavior Mario exhibits. Instead, it first exists *as* the relation between Mario and Jeanne, represented in the following sequence of turns:

Fragment 3
Mario: ((*fills goblet 4, counting out 6 yellow chips and then 12 red chips*))
Jeanne: [how many three dollars do you have?]
Mario: [how many three dollars do you have?] four

Jeanne's turn has two functions. It completes the first pair of turns, and, at the same time, it opens the second pair. The action of adding 12 chips thereby comes to be accounted for in terms of four groups of three chips. Here, the link between action and account exists between the two *as* social relation in the apparent connection of the query | reply pair: "how many three dollars do you have" | "fo[ur]." The two parts of this pair are query and reply because of their relation, because statements on their own have no function (Bakhtin, 1984). Thus, depending on the situation, the statement "four" can function as a question (as in turn 224), reply, order, acceptance of an order (e.g., when a waiter says, "four" after four individuals have asked for glasses of beer), invitation, acceptance of an invitation, and so forth. The function of "four" as a reply is a social relation (Vološinov, 1930). It is a reply because there is a query. To social, here, exists in the unity of the {query | reply} phenomenon. This phenomenon maps on a particular turn-taking sequence, where there first is a student action, then an invitation to produce an account, and then the acceptance of the invitation.

In their joint work, each part of which is intelligible and therefore social, Jeanne and Mario accomplish the mapping of a previous action and the contents of the goblet 4 onto mathematical representations that are treated as its equivalent. It is not as if there was something constructed that Mario now has to internalize. He not only and already participates in that practice. He also does the whole thing. But he does not do it on his own, alone. Instead, he does it together with the teacher, who already is familiar with the practice. It is *as* their relation that the practice first exists from the perspective of Mario, and it exists as relation again for the teacher, who, had she been asked to complete the task, would already

have exhibited it on her own. Jeanne here affirms the reply, and, in a way accessible to all, thereby acknowledges the appropriateness of what he has done in the context of mathematics. This acknowledgment (turns 225, 227) in fact allows Mario to discover that what he has done *is* the way of doing the work of generalizing. He now has a way of feeling what it means to generalize with respect to this particular task. He thereby is enabled to continue on his own, acting in a way that he has already done but together with the teacher. Now that he has done it on his own, we may ascribe the practice to him. Thus, what he now knows how to do already was present in the past, *as* the social relation. That is, the future (turns 228–229) was already present in the past (turns 218–220).

When Mario says that he understands, then something else manifests itself: the object/motive of the task. The goal of the task now has appeared in his consciousness. He now understands what is to go into each cell and how to produce it; and he uses the generalization, though it may be in the way young children know to produce grammatical sentences without actually knowing grammatical rules explicitly. He also *can* know and understand, though we have no evidence whether he actually does, the reasons for doing what he has done and not done earlier. As soon as the object reveals itself, in and especially *as* the exchange relation with Jeanne, he can and does orient towards it, as seen in the production of the contents of the remaining unfilled table cells. That object/motive reveals itself in and through the joint work of doing, which produces the object/motive. The structure of this event, therefore, is precisely the same as that in an example Vygotsky (1997b) provides for the emergence of the practice of pointing: First the child reaches out, then the mother hands him an object in the linear extension of the reach too short to grasp it, and then the child begins to point. The child's reach (or grasp) becomes a pointing in and through the mother's treatment of the movement as a pointing gesture. In the present instance, it is the social ratification of his (discursive) action in and through the teacher's treating it as an appropriate generalizing account.

Fragment 3 brings to the fore the fact that it is in and through the verbal statement that subjectivity is shared (see bracketed speech). That is, because language is a reality for two, any intrasubjectivity simultaneously is intersubjectivity. In fact, the specificity of the human relation is that the use of language, which externalizes subjective motivation for the other, is "capable of transforming the subjective behavioral motivation of the *other* into experience of the *self*—and the subjective behavioral motivation of the *self* into experience of the other" (Mikhailov, 2004, p. 27). In other words, there is the mutual transformation of experience of the self for the other, becoming shared experience, affectively shared community with the other. It is because of this shared experience that Mario and Jeanne can have any hope of moving the situation ahead, to get it unstuck, which will manifest itself also affectively in a positive way.

4. Discussion

In the preceding section, I provide an exemplifying account of the contradiction in learning activity. Students, by the very fact that they are to learn something, cannot actively orient towards the content of the learning (object). The preceding account exhibits not only when and how the object/motive of the activity reveals itself, but also how a mathematical practice that can be ascribed to a student in the future actually exists in the present: *as* the social relation of which the student is a constitutive part.

Readers somewhat familiar with cultural-historical approaches might want to state that the events exemplify (software, tool, teacher) *mediation* generally (e.g., Chassapis, 1998;

Mariotti, 2000; Zolkower & Shreyar, 2007) and *scaffolding* or *zone of proximal development* specifically (e.g., Goos, Galbraith, & Renshaw, 2002; Hussain, Monaghan, & Threlfall, 2013). I object to such explanations based on the fact that near the end of his life, Vygotsky was taking a theoretical turn (to Spinoza), which required him to abandon and critically review his own earlier work. Evidence for this has come to be known only recently, when his notebooks were found in the family archives (e.g., Zavershneva, 2010a, 2010b, 2010c).

First, past cultural-historical accounts of events such as the ones presented above use the notion of mediation. Thus, for example, one of the neo-Piagetian takes on the role of the teacher is as an intermediary between the child and the task (Figure 2a, 1). That is, the child does not relate directly to the task but via the teacher, who, different from child and task, connects the two (Grossen & Perret-Clermont, 1994). However, the same situation may also be considered, because of symmetry reasons, in terms of the mediational role of the task in the teacher–student relation (Figure 2a, 2). This time the teacher and student are considered as two independent entities, with their own subjectivities, and it is the task, external to both, that becomes the "middle entity" between and connecting the two. In another cultural-historical account, there are four independent entities (Cole & Engeström, 1993), here student, teacher, materials (goblet), and outcome (generalization) (Figure 2b). In the representation of the situation, one existing mediational relation exists between teacher, goblets, and generalization (solid lines, Figure 2b); the second mediational triangle links refers to the exchange, whereby the teacher mediates between student and goblets (solid lines, Figure 2b). The to-be-achieved mediational relation, the result of the exchange, is that between the student, the goblets, and the generalization (broken lines, Figure 2b).

It is true that Vygotsky (1989) has drawn such triangles in 1929, where the editors of his *Mind in Society* (Vygotsky, 1978) have taken it for their own, highly contested presentation.[3] Towards the end of his life, Vygotsky, turning for inspiration to Spinoza and his conception of one substance that manifests itself as both mind and body, is abandoning the notion of mediation (e.g., Mikhailov, 2001; Zavershneva, 2010a). In this context, Vygotsky (2001) develops the category *perežívanie* (experience), which he defines as the irreducible person–environment relation. This relation has pragmatic, intellectual, and affective colorings. If the relation is irreducible, this means that no part can be external to another; or, in other words, every part is part of all other parts. Rather than being separate entities (elements), as in the mediational models discussed above, no entity in the category stands on its own. Therefore, no part can stand *between* two other parts because they are already within and part of each other. *Mediation* no longer makes any sense. We see this in Fragment 3, which is an example of what Vygotsky previously thought to be a type of semiotic mediation.

[3] For the many problems with this edition of *Mind in Society* see Yasnitsky and van der Veer, 2016.

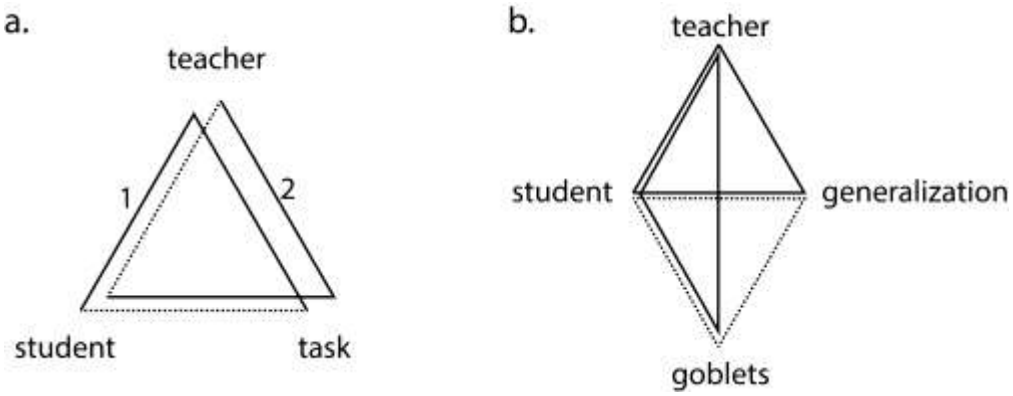

Figure 2. Two ways of theorizing mediation. a. A neo-Piagetian account of how intersubjectivity is "constructed" (Grossen & Perret-Clermont, 1994). b. Cultural-historical activity theory (Cole & Engeström, 1993)

In Fragment 3, I include what is reality for Mario, what exists for him: The words "how many three dollars do you have?", which are produced by the teacher's vocal cords, also ring in his ears. They are part of the already *shared* reality. As Vygotsky (1987) says, the word is impossible for one, but a reality for two persons. These words, therefore, do not stand *between* the teacher and student, but, instead, are *common and binding reality* for the two. The words constitute subjective consciousness of the teacher made objectively available, in the spoken words, to Mario and anyone else overhearing them. In a similar way, neither the goblets and their contents, nor the generalization mediate between other terms. First, the goblets are part of the environment of both; each participant is part of the (social) environment of the other. Second, and most importantly, the generalization exists *as* the social relation. That is, it is real, it exists for both of them, each contributing to the joint work that its production requires. The generalization, which constitutes the motive of the activity, is part of the shared reality rather than an entity that stands between, and thereby separating and bridging them.

Second, mathematics educators also use the notions of *scaffolding*, often together with that of the *zone of proximal development*. The latter is defined by the learner's independent problem solving and the actual performance with another, often more knowledgeable other (Vygotsky, 1978). Apart from the fact that the framing is completely oblivious to the fact that teachers also learn, while in verbal exchanges with students (e.g., Roth & Radford, 2010) both notions take us back to the mediational situation, where the teacher is theorized as standing between student and object, between student and cultural-historically established mathematical ideas (truths). Moreover, up to now, mathematics education research has neither distinguished between learning (quantitative-evolutionary) and development (qualitative, revolutionary), nor theorized the transition of quantitative-cumulative change into qualitative change that is associated with a change in consciousness—such as the appearance of the learning object in the consciousness of the learner.

How might we approach the transition from quantitative changes that occur when we do engage with a mathematical task into the changes of consciousness that result in and from activity? We can engage in activity only at our present level of development. How, then, can *new, qualitatively* different psychological forms arise, which, because they are of a different kind (the Mario who uses the generalization is a very different Mario than the one who does not), cannot accrue through cumulative quantitative change of what is already known? Vygotsky (1997a) draws on the dialectical law of the transformation of quantity into quality,

which accounts for morphogenesis, the emergence of new forms in a manner that is structurally equivalent to the mathematical formulation in catastrophe theory (Roth, 2016). Here, the current activity, which is associated with continuous change, precipitates a sudden turnover to the new form. Such activity, therefore, may be referred to as *leading activity*, that is, as activity that leads and leads to the transition to a qualitatively new form of consciousness. In the case study above, the task led to that kind of leading activity, whereas for Aurélie, it did not.

5. Implications for curriculum planning

In the preceding sections, I provide an event from a fourth-grade mathematics classroom and a discussion thereof in terms of a new direction indicated in the notebooks that Vygotsky wrote just before the end of his life. It shows how the motive of activity, which is presupposed in going constructivist and cultural-historical theories of learning, actually reveals itself in the course of the task. In fact, it is precisely when the learning outcome specified in the curriculum is achieved that the motive of the learning task *can* become available and exist for the learner. This creates a contradiction, which at the current time is not addressed in much of school learning.[4] Those readers with practical orientations might ask the question, but how do we address this paradox (contradiction) in actual classroom teaching? We already see in the present study one example, which rides heavily on the student–teacher exchange relation. But there are other ways of setting up situations, which provide us with different takes on the ways of dealing pragmatically with the learning paradox without making the futile attempt to making it disappear.

A first example comes from an experimental study conducted in 1933–1934 in the Palace for Pioneers in Kharkov (Leont'ev, 1982). In this study, youths were to build model aircraft. The organizers made available posters and instructors knowledgeable in the physics of flying. But the youths made only minimal use of these resources, orienting instead on make nice-looking artifacts. The psychologists involved in the project then proposed to change to motive of the task: Build models that will fly a specified distance. After the youths had built a first version, most models crashed prior to crossing the specified goal line. It was at this point that they began turning to posters and instructors, engaging in these exchanges many times longer than their predecessors had. In the process of finally having built a model that could fly the distance, students were becoming knowledgeable in the physics of flying. What had changed?

In this situation, the object/motive of the task was already available to the students, unlike in the above example from the mathematics classroom. For the first batch of youths, the object/motive was to build "nice" models; for the second batch, the object/motive was to build models that could fly a specified distance. That is, in both instances, the object/motive was real and grasped. In both instances, the youths could therefore establish whether the motive was realized in the current state of their model. When it became apparent to the second batch of youths that they were not getting closer to the intended outcome, they sought to expand their control over the conditions, which they did in seeking out knowledge concerning the physics of flying. Such an orientation that seeks to expand what can be done constitutes *expansive learning* (Holzkamp, 1993), and the expansion resulting from it has a

[4] In the 1980s, there has been a discussion of this phenomenon under the umbrella of the "learning paradox"; but it has never been solved to any satisfaction generally and in mathematics education specifically (e.g., Roth, 2012).

positive affective value—arising from the increased control and in the accomplishment of a task.

A second example comes from an observation I made while doing research related to science learning. While spending time outside of science in the participating classroom, the fifth-grade Jordan, who already had completed some other task, was asked to fill a mathematics worksheet with equivalent fractions of the type

$$\frac{a}{b} = \frac{c}{x}.$$

When he was done with the worksheet, I happened to be next to him. I asked to see the sheet and saw that he had every single item incorrect. I, then, offered the use of my laptop, where I loaded a version of MathCAD, a mathematical modeling program. I set it up showing an equivalent fraction, with the specific values for *a, b,* and *c* from his first item and asked, what number needed to be entered for *x* to have the computer respond whether it is correct or incorrect. I left Jordan to his own in attending to other students in the classroom. When I returned to him, he said, "I figured it out." Asked how to do the items, he said: "if the number on top is doubled, double the bottom; if the number is times three, make the bottom times three." Jordan had come across a generalization.

In this situation, there was an intelligible object: find the number in the missing spot that makes the two fractions equivalent. Before he had the computer, there was no way that Jordan could identify the quality of what he had produced. With the modeling software, feedback was provided that allowed him to identify in his own actions those that led to inappropriate and those that led to appropriate outcomes. As a result, he found the motive of the task, the generalization that produces equivalent functions, in his own actions. Once he had found it, he no longer needed the computer to provide feedback and he acted according to the accepted social practice. The function of the feedback was similar to the function of Vygotsky's mother giving the child the object out of reach, so that the hand/arm movement was treated as pointing with its own object/motive.

6. Coda

In this study, I take a cultural-historical approach to problematize the notion of the learning object, consciousness of which is required for the *intentional* engagement in mathematics tasks. I exemplify the learning paradox with a case study from a fourth-grade mathematics classroom and show how the learning object/motive arises as a result of activity, through the *joint* work of learner and teacher. I explicate why the theoretical terms of mediation does not make sense in the approach that the later Vygotsky was taking following his Spinozist turn, which required him to move mediation into the background and to the point of abolishing it. I offer examples intended to assist the designers of mathematics curriculum for developing tasks in which students can indeed conceive a motive, and where, seeking to expand their control over the task, students learn. In this case, both the expansion of agency (control over condition) and concrete realization of the motive are affectively reflected in a positive manner.

7. References

Bakhtin, M. M. (1984). *Problems of Dostoevsky's poetics*. Austin, TX: University of Texas Press.

Beswick, K., Watson, A., & de Geest, E. (2010). Comparing theoretical perspectives in describing mathematics departments: Complexity and activity. *Educational Studies in Mathematics, 75,* 153–170.

Chassapis, D. (1998). The mediation of tools in the development of formal mathematical concepts: ˇhe compass and the circle as an example. *Educational Studies in Mathematics, 37,* 275–293.

Cole, M. (1996). *Cultural psychology: A once and future discipline.* Cambridge, MA: Harvard University Press.

Cole, M., & Engeström, Y. (1993). A cultural historical approach to distributed cognition. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 1–46). Cambridge: Cambridge University Press.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics, 49,* 193–223.

Grossen, M., & Perret-Clermont, A.-N. (1994). Psychosocial perspective on cognitive development: Construction of adult-child intersubjectivity in logic tasks. In W. de Graaf & R. Maier (Eds.), *Sociogenesis reexamined* (pp. 243–260). New York, NY: Springer.

Holzkamp, K. (1983). *Grundlegung der Psychologie* [The founding of psychology]. Frankfurt/M, Germany: Campus.

Holzkamp, K. (1993): *Lernen: Subjektwissenschaftliche Grundlegung* [Learning: Foundation in a science of the subject]. Frankfurt/M, Germany: Campus.

Hussain, M. A., Monaghan, J., & Threlfall, J. (2013). Teacher-student development in mathematics classrooms: Interrelated zones of free movement and promoted actions. *Educational Studies in Mathematics, 82,* 285–302.

Husserl, E. (1973). *Husserliana Band I. Cartesianische Meditationen und Pariser Vorträge* [Husserliana vol. I. Cartesian meditations and Paris lectures]. The Hague, Netherlands: Martinus Nijhoff.

Il'enkov, E. V. (1977). *Dialectical logic: Essays on its history and theory*. Moscow, USSR: Progress.

Jaworski, B., & Potari, D. (2009). Bridging the macro- and micro-divide: Using an activity theory model to capture sociocultural complexity in mathematics teaching and its development. *Educational Studies in Mathematics, 72,* 219–236.

Kieran, C. (2002). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. *Educational Studies in Mathematics, 46,* 187–228.

Leont'ev, A. N. (1982). *Tätigkeit, Bewusstsein, Persönlichkeit* [Activity, consciounsness, personality]. Cologne, Germany: Pahl-Rugenstein.

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics, 44,* 25–53.

Marx, K., & Engels, F. (1962). *Werke Band 23* [Works vol. 23]. Berlin, Germany: Dietz.

Marx, K., & Engels, F. (1978). *Werke Band 3* [Works vol. 3]. Berlin, Germany: Dietz.

Mikhailov, F. T. (2001). The "other within" for the psychologist. *Journal of Russian and East European Psychology, 39*(1), 6–31.

Mikhailov, F. T. (2004). Object-oriented activity—Whose?. *Journal of Russian and East European Psychology, 42*(3), 6–34.

Rorty, R. (1989). *Contingency, irony, and solidarity*. Cambridge, UK: Cambridge University Press.

Roth, W.-M. (2011). *Passibility: At the limits of the constructivist metaphor*. Dordrecht, The Netherlands: Springer.

Roth, W.-M. (2012). Mathematical learning, the unseen and the unforeseen. *For the Learning of Mathematics, 32* (3), 15–21.

Roth, W.-M. (2016). *Concrete human psychology*. New York, NY: Routledge.

Roth, W.-M., & Lee, Y. J. (2007). "Vygotsky's neglected legacy": Cultural-historical activity theory. *Review of Educational Research, 77,* 186–232.

Roth, W.-M., & Radford, L. (2010). Re/thinking the zone of proximal development (symmetrically). *Mind, Culture, and Activity, 17,* 299–307.

Roth, W.-M., & Radford, L. (2011). *A cultural-historical perspective on mathematics teaching and learning*. Rotterdam, The Netherlands: Sense Publishers.

Roth, W.-M., & Walshaw, M. A. (2015). Rethinking affect in education from a societal-historical perspective: The case of mathematics anxiety. *Mind, Culture and Activity, 22,* 217–232.

Sheets-Johnstone, M. (2009). *The corporeal turn: An interdisciplinary reader*. Exeter, UK: Imprint Academic.

van Oers, B. (2001). Educational forms of initiation in mathematical culture. *Educational Studies in Mathematics, 46,* 59–85.

Vološinov, V. N. (1930). *Marksizm i folosofija jazyka: osnovye problemy sociologičeskogo metoda b nauke o jazyke* [Marxism and the philosophy of language: Main problems of the sociological method in linguistics]. Leningrad, USSR: Priboj.

Vygotskij, L. S. (2001). *Lekcii po pedologii* [Lectures on pedology]. Izhevsk, Russia: Udmurdskij University.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1987). *The collected works of L. S. Vygotsky, vol. 1: Problems of general psychology*. New York, NY: Springer.

Vygotsky, L. S. (1989). Concrete human psychology. *Soviet Psychology, 27*(2), 53–77.

Vygotsky, L. S. (1997a). *The collected works of L. S. Vygotsky vol. 3: Problems of the theory and history of psychology* (R. W. Rieber & J. Wollock, Eds.). New York, NY: Plenum.

Vygotsky, L. S. (1997b). *The collected works of L. S. Vygotsky, vol. 4: The history of the development of higher mental functions*. New York, NY: Springer.

Vygotsky, L. S. (1998). *The collected works of L. S. Vygotsky vol. 5: Child psychology*. New York, NY: Kluwer Academic/Plenum Publishers.

Zavershneva, E. Iu. (2010a). The Vygotsky family archive (1912–1934): New findings. *Journal of Russian and East European Psychology, 48*(1), 14–33.

Zavershneva, E. Iu. (2010b). The Vygotsky family archive: New findings. *Journal of Russian and East European Psychology, 48*(1), 34–60.

Zavershneva, E. Iu. (2010c). The way to freedom. *Journal of Russian and East European Psychology, 48*(1), 61–90.

Yasnitsky, A., & van der Veer, R. (2016). *Revisionist revolution in Vygotsky studies: The state of the Art.* East Sussex, UK: Routledge.

Zolkower, B., & Shreyar, S. (2007). A teacher's mediation of a thinking-aloud discussion in a 6th grade mathematics classroom. *Educational Studies in Mathematics, 65*, 177–202.